A Theory of Satisfiability-Preserving Proofs in SAT Solving

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Abstract

We study the semantics of propositional interference-based proof systems such as DRAT and DPR. These are characterized by modifying a CNF formula in ways that preserve satisfiability but not necessarily logical truth. We propose an extension of propositional logic called *overwrite logic* with a new construct which captures the meta-level reasoning behind interferences. We analyze this new logic from the point of view of expressivity and complexity, showing that while greater expressivity is achieved, the satisfiability problem for overwrite logic is essentially as hard as SAT, and can be reduced in a way that is well-behaved for modern SAT solvers. We also show that DRAT and DPR proofs can be seen as overwrite logic proofs which preserve logical truth. This much stronger invariant than the mere satisfiability preservation maintained by the traditional view gives us better understanding on these practically important proof systems. Finally, we showcase this better understanding by finding intrinsic limitations in interference-based proof systems.

1 Introduction

The story of SAT solving is one of great success, which has made of SAT solvers widely used in practical applications due to their ability to routinely solve instances with millions of variables. Two recent, interrelated breakthroughs in SAT solving are the extension of conflict-driven clause learning (CDCL) solvers [28] with inprocessing techniques that allow efficient solving in fragments where CDCL has exponential behavior [1, 24, 19]; and the introduction of increasingly expressive and easy to check proof systems to certify the correctness of unsatisfiability results [7, 11, 41, 17].

The connection between these two features is subtle, and it is yet to be fully understood how to exploit it to improve the solvers' efficiency. Inprocessing techniques [19], including clause elimination [10], symmetry breaking [1], bounded variable addition [24], and parity reasoning [35], often tackle fragments hard for CDCL [37, 38, 9]. Behind the scene, these techniques briefly violate the rules of the proof system underlying SAT solvers, namely DAG-like resolution [2], in a sound way. Escaping the limitations of this proof system also allows to circumvent their known pitfalls in terms of complexity. For example, symmetry breaking or bounded variable addition allow to derive clauses that are not implied by the formula, but rather just consistent with it. In doing so, potential exponential speed-ups are attained [12, 17], and the solver briefly operates under much more general implicit proof systems.

The delete resolution asymmetric tautology (DRAT) and delete propagation redundancy (DPR) proof systems were developed to express and check these operations [41, 17]. While DRAT proof generation is widely supported [25, 29, 13], DPR is fairly recent and is yet to be adopted in practice. Recent results show that these proof systems are polynomially equivalent to the extended resolution proof system [16, 20], for which no exponential lower bounds are

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known [22]. In other words: if SAT solvers fully exploited the power of DRAT and DPR, no known fragments of CNF formulas would be hard to solve. Alas, this is far from being the case, despite some recent advances in exploiting DPR-based reasoning [18], and constructing methods that capitalize on DPR is notably complex due to the involvement of the whole formula in the inferences, a phenomenon known as *interference* [14].

Interference has practical implications beyond SAT solving, especially when one considers the use of proofs with aims other than checking the solvers' results. For example, no method exists in the literature to generate Craig interpolants [5] from DRAT proofs that can be used in model checking [27], and the allowed inferences lack of some of the features familiar and intuitive from other proof systems such as resolution [30]. The issues of interference all boil down to the allowed inferences being satisfiability-preserving, rather than just truth-preserving [30]. For example, methods to obtain interpolants from other proof systems work by recursively constructing partial interpolants throughout the proof tree [26, 40, 8, 34], but the invariants ensuring the correctness of this procedure rely strongly on the proof being truth-preserving, i.e. the conclusion of every inference being a consequence of its premises [30]. Even worse, DRAT and DPR proofs are not even tree-shaped. Rather, these proofs modify the formula in an incremental way by introducing and deleting clauses. Again, this is mandated by interference: since the *absence* of some clauses is a requirement for some inferences, proofs in the shape of clause trees do not make much sense. It has been argued that further research is needed to fully understand the potential and possible shortcomings of interference reasoning [14].

Contributions One way to understand DPR proofs is to find what are the semantic invariants preserved throughout proofs. In truth-preserving proof systems this is straightforward, but previous results imply that no such invariants exist for DPR proofs [30]. In this paper, we argue that DPR can be construed as operating over a more general logic, called *overwrite logic*, which derives expressions that generalize clauses. Reasoning under this perspective is similar to the assumptions without loss of generality that are familiar in mathematics. Not only can we find stronger invariants when using overwrite logic: DPR proofs, which are satisfiability-preserving proofs when considered over propositional logic, become truth-preserving proofs. Moreover, we argue that for any practical application overwrite logic is no more different from CNF formulas than propositional logic, and Tseitin-like procedures [31] exist with similar complexity. Finally, this new perspective allows inferences that cannot be performed with interference reasoning.

Related work DRAT and DPR proofs [41, 17] were developed to increase the reliability of SAT solvers' unsatisfiability results by allowing powerful inferences that easily expressed the reasoning techniques used by SAT solvers [25, 13, 29]. Recent work shows that both proof systems are essentially as expressive as extended resolution [16, 20], for which no exponential lower bounds exist [22]. Some early work on interference-based solving techniques exists but is still relatively limited [18]; a goal of this work is to provide a deeper understanding of interference that furthers the development of such techniques.

The issue of the semantics preserved by DRAT proofs has been tackled in [30] for a restricted case without deletion, presenting an intuitive explanation in terms of permissive definitions. A result presented there seemed to prevent semantic invariants stronger than satisfiability preservation in the unrestricted case; we show that overwrite logic is able to overcome this limitation. The model extraction method from [19] bears some clear similarities to our work, although the focus there is on identifying correctness criteria for interference during solving. Some inconsistencies on DRAT proof checking have their roots on the unexpected interaction between deletion and interference reasoning [33].

Many methods for extracting interpolants from resolution-like proofs have been proposed [26, 40, 8, 34], but truth-preservation in inferences is required. (Propositional) interpolant extraction from extended resolution proofs, hence from DRAT proofs, has been shown to be exponential under cryptographic assumptions [3, 23]. Since many model checking approaches are interpolation-based [27], or can work as interpolant generators [4], there are compelling reasons to study the interaction between interpolation and interference reasoning.

2 Preliminaries

We consider a countable set of propositional variables. An interpretation maps each propositional variable to either 0 or 1. Throughout this paper we define several kinds of logical expressions, and semantics for them. For each logical expression E, we give a notion of when an interpretation I satisfies E; in this case we write $I \vDash E$. An expression E is satisfiable whenever there is some interpretation I with $I \vDash E$, and unsatisfiable otherwise. Furthermore, given two logical expressions E_1, E_2, we say that E_1 entails E_2 whenever $I \vDash E_2$ for every interpretation I such that $I \vDash E_1$; and that E_1 satisfiability-entails E_2 whenever the existence of an interpretation I_1 with $I_1 \vDash E_1$ implies the existence of an interpretation I_2 such that $I_2 \vDash E_2$. We denote this by $E_1 \vDash E_2$ and $E_1 \vDash_{\text{sat}} E_2$, respectively. We say that E_1 is equivalent to E_2 whenever $E_1 \vDash E_2$ and $E_2 \vDash E_1$; and that E_1 is satisfiability-equivalent to E_2 whenever $E_1 \vDash_{\text{sat}} E_2$ and $E_2 \vDash_{\text{sat}} E_1$. We denote this by $E_1 \equiv E_2$ and $E_1 \equiv_{\text{sat}} E_2$, respectively. The set of variables occurring in an expression E is denoted by $\text{Var}(E)$

In this paper we consider CNF formulas and propositional formulas as expressions of different logics, in order to clearly compare them and their variants from a complexity perspective. Let us first start with propositional logic. We consider propositional logic (PL) formulas as recursively defined by the following Backus-Naur form:

$$\varphi \coloneqq \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$$

where p represents propositional variables. The semantics are defined as usual, e.g. $I \vDash p$ whenever $I(p) = 1$, and $I \vDash \varphi \vee \psi$ whenever $I \vDash \varphi$ or $I \vDash \psi$.

We now define clausal normal form (CNF) logic. A literal is an expression of the form $+p$ or $-p$, where p is a propositional variable[1]. The complement of a literal is defined by $\overline{+p} = -p$ and $\overline{-p} = +p$; we then say that $+p$ and $-p$ are complementary literals. A cube is an expression of the form $\langle l_1, \ldots, l_n \rangle$, and a clause one of the form $[l_1, \ldots, l_n]$, where the l_i form a finite, complement-free set of unique literals. We regard both cubes and clauses as sets of literals. The complements of clauses and cubes are given by $\overline{\langle l_1, \ldots, l_n \rangle} = [\overline{l_1}, \ldots, \overline{l_n}]$ and $\overline{[l_1, \ldots, l_n]} = \langle \overline{l_1}, \ldots, \overline{l_n} \rangle$. A CNF formula is a finite set of clauses. The semantics of CNF logic are given as usual: the literal $+p$ is equivalent to p, and $-p$ is equivalent to $\neg p$; clauses (resp. cubes) are interpreted as disjunctions (resp. conjunctions) of their literals; and CNF formulas are interpreted as conjunctions of their clauses. Given a clause C and a cube Q such that $Q \nvDash C$, i.e. Q and C have no literals in common, then we define the clause $C|_Q$ as $C \setminus \overline{Q}$. We lift this definition to CNF formulas, so $F|_Q = \left\{ C|_Q \mid C \in F \text{ and } Q \nvDash C \right\}$.

[1] Albeit unusual, the distinction between literals and variables or their negations makes somewhat easier the definition overwrites later in Section 3; it does not otherwise affect the content.

2.1 Redundancy notions on CNF logic

For the rest of this section, we focus on CNF logic, since it is the logic modern SAT solvers operate within. They work by iteratively modifying a CNF formula in a satisfiability-equivalent way [10, 41]. In particular, redundant clauses are introduced into or deleted from the formula. A clause C is said to be redundant in a formula F whenever $F \cup \{C\} \equiv_{\text{sat}} F$. Several refinements of the notion of redundancy have been proposed, and new inference techniques arise from them.

Entailment criteria It is straightforward to observe that C is redundant in F whenever $F \vDash C$, so entailment criteria are a good source of redundancy criteria. Checking whether C is entailed by F is coNP-complete [18], but more restricted criteria are relatively easy to check. Two simple entailment criteria can be combined to express every entailed clause [21]. The first one is resolution: given two clauses C and D, and a literal l, we say that C is resolvable with D upon l (or simply that the resolvent $C \otimes_l D$ exists) whenever l is the only literal k such that $k \in C$ and $\overline{k} \in D$. In this case, we call the clause $C \otimes_l D$ defined by $(C \setminus \{l\}) \vee (D \setminus \{\overline{l}\})$ the resolvent of C with D upon l; it can be then shown that $\{C, D\} \vDash C \otimes D$. Since the literal l is unique, we will sometimes omit it and simply write $C \otimes D$. The second criterion is subsumption: we write $C \sqsubseteq D$ whenever every literal in C is contained in D; in that case, we have $C \vDash D$.

A particular combination of resolution and subsumption yields an entailment criterion able to express clauses introduced in CDCL SAT solving [6, 2], called reverse unit propagation (RUP) [7]. Its particular definition is not relevant for this work; we briefly remark that if a clause C is a RUP clause in F, then $F \vDash C$. We refer the reader to [30] for a detailed discussion.

Interference criteria More generally, redundancy criteria exist for which the redundant clause is not entailed by the formula, but rather is simply consistent with it. In such criteria, both the presence *and the absence* of some clauses are required; this phenomenon is known as *interference* [14]. Several interrelated interference criteria have been proposed. Resolution asymmetric tautologies (RAT) [10, 41] are widely used, but so far the most powerful criterion, subsuming all others to the best of our knowledge, is that of propagation redundancy (PR) [17].

A clause C is a PR clause in F upon a cube Q whenever, $Q \cap C$ is nonempty, and furthermore every clause in $F|_Q$ is a RUP in the CNF formula $F|_{\overline{C}}$. In this case, it can be shown that $F \equiv_{\text{sat}} F \cup \{C\}$ [17]. From a model-theoretical standpoint, the condition that such clauses must be RUPs is quite restrictive; in fact, the satisfiability-equivalence above holds if "RUP" is replaced by any other entailment criterion, including entailment itself. PR clauses are very powerful: every nonempty redundant clause can be expressed as a PR clause [17]; unfortunately, finding the appropriate cube Q is NP-complete [18].

2.2 Interference-based proofs

Traditional proof systems represent proofs as trees (or, if clauses are repeated, as DAGs) with clauses as nodes and edges representing truth-preserving inferences, i.e. inferences where the conclusion is entailed by the premises. However, this approach is inadequate for interference-based inferences, since a representation of the "clauses derived so far" must be kept. In general, satisfiability-preserving inferences preclude tree-shaped proofs, since they violate monotonicity: if C is redundant in F, it does not follow that C is redundant in $F \cup G$, whereas this property holds if entailment is considered instead of redundancy [30].

There are compelling reasons to use satisfiability-preserving inferences. From a theoretical perspective, satisfiability-preserving inferences allow exponentially shorter proofs [22]. A more pragmatic standpoint also shows that proof trees are not appropriate: proofs generated by SAT

solvers follow the solver run, deriving essentially the same clauses [11]. However, SAT solvers frequently delete redundant clauses [10, 11], and since we do not have monotonicity, deletion can *enable* satisfiability-preserving inferences, which would never happen with truth-preserving inferences. This makes it hard to reason about interference criteria, to the extent that the interplay between interference and clause deletion is at the root of some discrepancies between the theory and the practice of proof logging in SAT solvers [33]. Hence, for interference-based proofs, an instruction string approach is more appropriate.

Instructions are expressions which come in two flavors: introductions $\mathbf{i:}\, C$, deletions $\mathbf{d:}\, C$, where C is a clause. Optionally, introduction instructions can be annotated by a cube Q as in $\mathbf{i:}\, Q \blacktriangleright C$. Each instruction string maps a CNF formula F into its *accumulated CNF formula*, also called *effect*, as follows:

$$\mathrm{eff}_F(\epsilon) = F \qquad \mathrm{eff}_F(\mathbf{i:}\, C,\, \pi) = \mathrm{eff}_{F\cup\{C\}}(\pi) \qquad \mathrm{eff}_F(\mathbf{d:}\, C,\, \pi) = \mathrm{eff}_{F\setminus\{C\}}(\pi)$$

Intuitively, the accumulated formula applies the instructions along the string: $\mathbf{i:}\, C$ introduces C in the CNF formula, and $\mathbf{d:}\, C$ deletes C from it.

With the purpose of generating unsatisfiability certificates that can be both efficiently generated and efficiently checked, the *Delete Resolution Asymmetric Tautology* (DRAT) and *Delete Propagation Redundant* (DPR) proof systems were introduced [41, 17]. State-of-the-art SAT solvers are able to generate a DRAT proof when reporting an unsatisfiable CNF formula. Its extension to DPR proofs was developed recently, and allows for practically shorter and easier to generate proofs than DRAT [17]; theoretically, both proof systems are (polynomially) as powerful as extended resolution, though [16, 20]. The DPR proof system allows introduction and deletion instructions under the following conditions:

- ϵ is a DPR derivation from F.
- $\mathbf{i:}\, C,\, \pi$ is a DPR derivation from F if π is a DPR derivation from F, and C is a RUP clause in $\mathrm{eff}_F(\pi)$.
- $\mathbf{i:}\, Q \blacktriangleright C,\, \pi$ is a DPR derivation from F if π is a DPR derivation from F, and C is a PR clause in $\mathrm{eff}_F(\pi)$ upon Q.
- $\mathbf{d:}\, C,\, \pi$ is a DPR derivation from F whenever π is a DPR derivation from F.

The DRAT proof system can be obtained by further requiring the annotation Q in the third condition to have exactly one literal l: the clause C would in that case be RAT in $\mathrm{eff}_F(\pi)$ upon l [17]. An DPR refutation of a CNF formula F is an DPR derivation π from F such that $[] \in \mathrm{eff}_F(\pi)$. If π is a DPR derivation from a CNF formula F, then $F \vDash_{\mathrm{sat}} \mathrm{eff}_F(\pi)$; in particular, if it is a refutation of F, then F is unsatisfiable.

3 Overwrite propositional logic

The proof of correctness of PR clause introduction in [17] relies on modifying an interpretation by *overwriting* variable truth assignments. It is thus apparent that assignment overwriting plays a pivotal role in the semantics of DPR proofs. This was already crucial in the approach from [30], where a variation on the form of formulas is explored. Unfortunately, the definitional formulas discussed in that work are cumbersome to work with and not versatile enough to deal with deletion. We propose a more natural approach which only adds one constructor to propositional logic; we call the obtained logic *overwrite propositional logic* (OPL).

Given an interpretation I and a cube B, the overwrite of B on I is the interpretation $I + B$ defined by:

$$(I + B)(p) = I(p), \text{ if } p \notin \mathrm{Var}(B) \quad (I + B)(p) = 0, \text{ if } -p \in B \quad (I + B)(p) = 1, \text{ if } +p \in B$$

The interpretation $I + B$ is well-defined, because B does not contain complementary literals. Intuitively, the interpretation $I + B$ is obtained by minimally forcing I to satisfy B.

We define a rule as an expression of the form $(B :\!- E)$, where B is a cube and E is any logical expression. Rules are not logical expressions, i.e. they are merely syntactic building blocks and do not have truth values. Given an interpretation I, the conditional overwrite of B on I subject to E is the interpretation $I + (B :\!- E)$ defined by:

$$I + (B :\!- E) = I + B, \text{ if } I \vDash E \qquad\qquad I + (B :\!- E) = I, \text{ if } I \nvDash E$$

Starting from propositional logic, *overwrite propositional logic* (OPL) is obtained by considering one extra connective: given two OPL formulas φ and ψ, and a cube B, the expression $\nabla(B :\!- \psi).\, \varphi$ is an OPL formula. Its semantics are given by conditional overwrites: $I \vDash \nabla(B :\!- \psi).\, \varphi$ whenever $I + (B :\!- \psi) \vDash \varphi$.

In the following, we consider strings of rules $\vec{\varepsilon} = \varepsilon_1 \dots \varepsilon_n$, where the ε_i are rules. We denote by $\nabla\vec{\varepsilon}.\, E$ the iterative application of overwrite operators to the expression E, i.e. $\nabla\varepsilon_1.\, \nabla\varepsilon_2.\, \dots \nabla\varepsilon_n.\, E$. Similarly, we denote by $I + \vec{\varepsilon}$ an iterative conditional overwrite on I, i.e. $I + \varepsilon_1 + \varepsilon_2 + \dots \varepsilon_n$.

Clause-based normal forms As for propositional logic, clause-based normal forms can be defined for overwrite propositional logic. An initial restriction can be made on the form of rules. Whereas OPL allows rules of the form $(B :\!- \varphi)$, where φ is an arbitrary OPL formula, we will show that it suffices to consider *cubic rules* given by $(B :\!- Q)$, where Q is a cube. Overwrite clausal normal form (OCNF) is obtained by stacking overwrite connectives with cubic rules over each clause. Another restriction can be obtained by writing a stack of overwrite connectives over a CNF formula; we call such expressions uniformly overwrite clausal normal forms (UOCNF). As we show in Section 3.2, the expressive power of OPL, OCNF and UOCNF is polynomially the same; this is unlike the case for propositional logic, where PL is exponentially more expressive than CNF.

An overwrite clause is an expression of the form $\nabla\vec{\varepsilon}.\, C$ where $\vec{\varepsilon}$ is a (possibly empty) string of cubic rules, and C is a clause; an OCNF formula is a set of overwrite clauses. The semantics are given as expected: I satisfies the overwrite clause $\nabla\vec{\varepsilon}.\, C$ whenever $I + \vec{\varepsilon}$ satisfies C; and I satisfies an OCNF formula F whenever I satisfies every overwrite clause in F. Furthermore, if F is a (standard) CNF formula, then we say that $\nabla\vec{\varepsilon}.\, F$ is a UOCNF formula, again with the expected semantics: $I \vDash \nabla\vec{\varepsilon}.\, F$ whenever $I + \vec{\varepsilon} \vDash F$. The following result shows that UOCNF can be naturally embedded into (i.e. regarded as a sub-logic of) OCNF:

Proposition 1. *$\nabla(B :\!- \psi)$ distributes with $\vee$, $\wedge$ and $\neg$. In particular, $\nabla\vec{\varepsilon}.F \equiv \{\nabla\vec{\varepsilon}.\, C \mid C \in F\}$.*

Expressivity and complexity of overwrite formulas Having extended propositional logic PL with the overwrite connective to obtain the new logic OPL and its restrictions OCNF and UOCNF, it is natural to ask how do their expressivities compare, and what is the complexity of deciding their satisfiability. Expressivity can be studied from a *qualitative* as well as *quantitative* point of view. For instance, it is well known that for every CNF formula an equivalent PL formula exists and *vice versa* [39], so we say that PL and CNF have the same qualitative

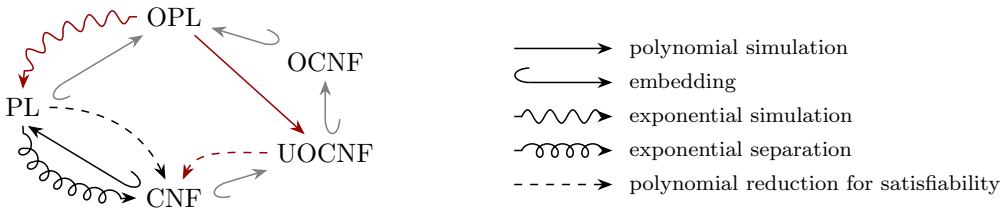

Figure 1: Simulation landscape between the CNF, PL, OPL, OCNF and UOCNF logics.

expressivity. However, CNF formulas need in the worst case to be exponentially larger than some equivalent PL formulas [39]; we thus say that PL is quantitatively more expressive than CNF.

When we talk about expressivity we consider the existence of *truth-equivalent* formulas. A different question is the complexity of the satisfiability problem for a given formula in a logic, in particular whether a polynomially-sized, *satisfiability-equivalent* formula can be found in polynomial time. Following our example, even if PL is quantitatively more expressive than CNF, the Tseitin procedure [31] provides a way to reduce the satisfiability problem for PL to the satisfiability problem for CNF in linear time, and so both satisifiability problems are NP-complete.

Figure 1 displays what we will know by the end of this section. For example, three arrows are shown between PL and CNF. The inclusion arrow shows that CNF can be embedded in PL; the coiled arrow shows that transforming PL formulas into equivalent CNF formulas is worst-case exponential; and the dashed arrow shows that the satisfiability problem for PL can be polynomially reduced to the satisfiability problem for CNF. These simulations are shown in black because they are known from previous work. Simulations in grey are straightforward, namely the embeddings CNF $\subseteq$ PL $\subseteq$ OPL, and CNF $\subseteq$ UOCNF $\subseteq$ OCNF $\subseteq$ OPL. Finally, three non-trivial simulations are shown in red, so we explain them in the next sections. All in all, by the end of this section we will have argued the following results:

1. CNF, PL, UOCNF, OCNF and OPL all have the same qualitative expressivity.
2. UOCNF, OCNF and OPL all have the same quantitative expressivity.
3. The procedure we provide to simulate OPL through PL is worst-case exponential.
4. Despite the latter two results, the satisfiability problem for OPL can be reduced to the satisfiability problem for CNF in a manner suitable for SAT solving.

3.1 Qualitative expressivity

In the following paragraphs we argue that OPL, OCNF, UOCNF, PL and CNF have all the same qualitative expressivity, i.e. for every formula in one of these logics, we can find a truth-equivalent formula in any other of them. This does not a priori need to be the case, for there exist sets of interpretations that are not expressible within PL, e.g. the set of interpretations that map exactly one variable to 1. We show this by finding a cycle of simulations through all of these logics. The embeddings CNF $\subseteq$ UOCNF $\subseteq$ OCNF $\subseteq$ OPL, together with the well-known (exponential) transformation of PL formulas into equivalent CNF formulas [39], leaves only the simulation of OPL by PL to be shown. We devote the rest of this section to obtain a procedure that, given an OPL formula, generates a truth-equivalent PL formula.

Let us first consider PL formulas ψ and φ, so they do not contain ∇ connectives. One can show the following equivalence:

$$\nabla(B :- \psi).\, \varphi \equiv (\psi \vee \varphi) \wedge (\neg\psi \vee \varphi|_B) \tag{1}$$

where $\varphi|_B$ is obtained by replacing variables x in φ by $\top$ if $+x \in B$ and by $\bot$ if $-x \in B$. Observe that the PL formula $\varphi|_B$ is truth-equivalent to the CNF formula $F|_B$ as defined in Section 2 whenever F is a CNF formula truth-equivalent to φ, so the notation is consistent; for consistency too, we call $\varphi|_B$ the reduct of φ under B. Formula (1) expresses the intuition that either the condition ψ holds, in which case we must evaluate φ with respect to the interpretation overwritten by B, or ψ does not hold and we evaluate φ as usual.

In order to extend this transformation to arbitrary OPL formulas, we need to generalize also the notion of reduct to OPL formulas. However, this is not so straightforward, since it raises the question how would one replace literals inside overwrite rules by $\top$ or $\bot$. We instead generalize the notion of reduct in such a way that it directly yields a PL formula, which effectively defines a truth-equivalent transformation from OPL to PL. In particular, we define:

$$(\nabla(B :- \psi).\, \varphi)|_A = (\psi\,|_A \vee \varphi|_A) \wedge (\neg\psi\,|_A \vee \varphi|_{B \cup A \setminus \overline{B}})$$

Before we continue, let us justify this definition, which may seem odd at first. In order to compute this reduct, we first apply the transformation from (1) to the formula $\nabla(B :- \psi).\, \varphi$, and then apply the reduct using the definitions for the other connectives. We then obtain:

$$(\psi\,|_A \vee \varphi|_A) \wedge (\neg\psi\,|_A \vee \varphi\,|_B\,|_A)$$

We then observe that $\varphi\,|_B\,|_A = \varphi|_{B \cup A \setminus \overline{B}}$ holds for every propositional formula. Having clarified this point, we now show a result characterizing the semantics of reducts, and we obtain as an immediate consequence that PL qualitatively simulates OPL.

Theorem 1. *Let φ be an OPL formula and A be a cube. For every interpretation I, we have $I \vDash \varphi|_A$ if and only if $I + A \vDash \varphi$. In particular, the PL formula $\varphi|_\emptyset$ is truth-equivalent to φ.*

3.2 Quantitative expressivity

We now analyze the question about the relative quantitative expressiveness of each logic. It is known that PL is exponentially more expressive than CNF. This in particular means that there exists no mapping Φ that maps PL formulas φ to truth-equivalent CNF formulas $\Phi(\varphi)$ such that the size of $\Phi(\varphi)$ is polynomially-sized on the size of φ.

Remarkably, a similar separation does not exist between OPL formulas and neither of their clause-based normal forms OCNF and UOCNF. In fact, any OPL formula can be linearly transformed into a *truth-equivalent* UOCNF formula. The Tseitin transformation achieves a similar result from PL formulas into CNF formulas, but in this case the transformation is only satisfiability-preserving, albeit some additional features are preserved. Together with the embeddings UOCNF $\subseteq$ OCNF $\subseteq$ OPL, the quantitative equivalence of these three logics follows.

Our procedure is much similar to the Tseitin transformation, with two main differences. On the one hand, the presence of an overwrite prefix in an UOCNF allows us to circumvent the loss of truth preservation: values are assigned to Tseitin variables using overwrites. On the other hand, an additional transformation for the overwrite connective is necessary.

The Tseitin transformation can be understood as follows. First, a formula is recursively transformed into a Boolean circuit, possibly with sharing, and such a circuit does not contain

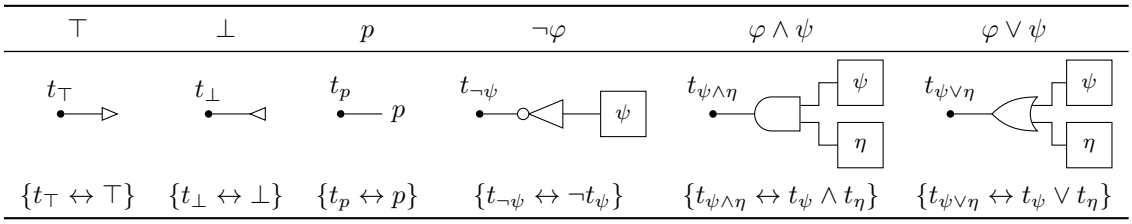

Figure 2: Tseitin transformations of subformulas into circuits and definitions. Variables t_φ represent the circuit output for subformulas φ.

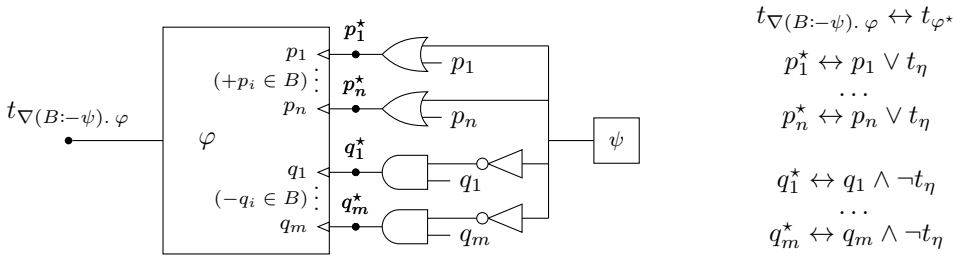

Figure 3: Circuit associated to the formula $\nabla(B :- \psi).\,\varphi$. Triangles represent the use of the outputs $p_i^\star, q_i^\star$ as the inputs p_i, q_i in the ψ circuit. Introduced definitions are shown in the right: variables $p_i^\star, q_i^\star$ are required to be new.

cycles. Then, new variables are introduced for every node in the circuit. For every such new variable x, connected through a gate to nodes whose input nodes have variables $y_1, \ldots, y_n$, a definition of the form $x \leftrightarrow \varphi(y_1, \ldots, y_n)$ of bounded size is generated, as shown in Figure 2.

Our procedure completes the Tseitin transformation by providing transformation rules to turn the overwrite connective $\nabla(B :- \psi).\,\varphi$ into a circuit and a set of definitions. Let us split B into its positive literals $+p_1, \ldots, +p_n$ and its negative literals $-q_1, \ldots, q_m$. Given circuits for the subformulas φ and ψ, we construct the circuit shown in Figure 3. The circuit for φ has as inputs the variables occurring in φ, which may or may not include variables from $p_1, \ldots, p_n, q_1, \ldots, q_m$. Now, for any interpretation I, the formula φ is evaluated under the interpretation $I + (B :- \psi)$, and so the variables p_i take the new truth value 1 if either ψ or p_i are satisfied by I; and the variables q_i take the new truth value 1 if q_i is satisfied and ψ is falsified by I. Hence, the circuit computes new auxiliary variables $p_1^\star, \ldots, p_n^\star, q_1^\star, \ldots, q_m^\star$ that express the values of the modified variables after the application of the rule $(B :- \psi)$, and then uses the starred variables as the input of φ. If a variable does not occur in B, then that variable is itself used as input in φ.

In order to encode each generated subcircuit, definitions as shown in Figures 2 and 3 can be generated and collected in a set D. Observe that, for every node named with x, a unique definition of the form $x \leftrightarrow \psi$ exists in D. Furthermore, since the circuit is acyclic, the nodes can be topologically sorted as $x_1, \ldots, x_n$, i.e. if there is a path from x_i to x_j in the circuit, then $i < j$. Now, let us clausify the definition $x_i \leftrightarrow \varphi_i$ for every node x_i, obtaining the CNF formula H_i, whose size is bounded by a constant. Each obtained clause $C \in H$ contains either $+x$ or $-x$, so it can then be turned into the rule $\big(\{\pm x_i\} :- \overline{C}\big)$, where the polarity of x_i is chosen to be the same as the occurring in C. Then, we define a string $\vec{\varepsilon_i}$ given by the concatenation of all such rules for the definition $x_i \leftrightarrow \varphi_i$. The following result then holds:

Theorem 2. *In the conditions above, $\varphi \equiv \nabla\varepsilon_1 \ldots \varepsilon_n.\{[+t_\varphi]\}$, where t_φ is the variable associated*

to the output of the constructed circuit.

Let us briefly explain the intuition behind Theorem 2. By induction, one can show that, after applying the rule string $\vec{\varepsilon}_1 \dots \vec{\varepsilon}_i$, the variables $x_1, \dots, x_i$ get assigned the respective truth values that the nodes labeled by $x_1, \dots, x_i$ would get in the circuit; in particular, the node $x_n = t_\varphi$ gets assigned the truth value of the circuit output, which is equivalent to φ.

This finishes the proof that OPL, UOCNF and OCNF are all equally expressive. Furthermore, given the embedding PL $\subseteq$ OPL, and that PL is exponentially more expressive than CNF, we know that there is an exponential separation between CNF and OPL. However, the question whether OPL is exponentially more expressive than PL remains. In general, the translation from Section 3.1 is exponential:

Example 1. Consider the sequence of OPL formulas $\varphi_n = \nabla(+x_n :\!- x_n - 1) \dots (+x_1 :\!- x_0).\bot$ for $n \in \mathbb{N}$. Given any cube B which does not refer to variables $x_0, \dots, x_n$, one can show by induction that $\varphi_{n+1}|_B = \big(x_n \vee \varphi_n|_\emptyset\big) \wedge \big(\neg x_n \vee \varphi_n|_\emptyset\big)$. This means that $\big|\varphi_n|_\emptyset\big|$ is exponential on n, which implies that our transformation from OPL formulas to PL formulas is worst-case exponential. ■

From our argument throughout this section, it becomes apparent that the expressivity of OPL is similar to that of (sharing) circuits. To the best of our knowledge, it is an open question whether an exponential separation exists between circuits and propositional formulas, so we speculate the following:

Conjecture 1. *OPL is exponentially more expressive than PL.*

3.3 Complexity of the satisfiability problem

Any potential use of overwrite logic in practice would be hindered if new solvers for this logic would need to be developed. In particular, there are two closely related questions that must be answered: 1) what is the complexity of the satisfiability problem for OPL, and 2) whether efficient and effective translations into existing solving paradigms. In Section 3.2 we have shown that an OPL formula can be linearly translated into an equivalent circuit. Linear encodings of circuits into *almost* truth-equivalent formulas through the Tseitin procedure and variations thereof are well known [31], in the sense that all interpretations of each formula coincide in the variables occurring in the OPL formula. In a more formal way, the definitions from Figures 2 and 3, together with an assertion of the output node, is a suitable translation to CNF in this sense. We then conclude:

Corollary 1. *The satisfiability problem for OPL formulas is NP-complete. Furthermore, every OPL formula φ can be linearly translated into a CNF formula F such that $F \vDash \varphi$, and for every interpretation I satisfying φ another interpretation J satisfying F exists such that I and J coincide on* $\mathrm{Var}(\varphi)$.

4 Understanding DPR

Throughout this section, we will obtain results that clarify the semantics of DPR proofs, i.e. the semantic invariants they preserve. However, before we get to that, we believe it is useful to understand DPR proofs in terms of a technique widely used in handwritten proofs, namely making assumptions without loss of generality. We now provide two examples of use of these techniques; later in this section we will explain their relation to DPR proofs.

Example 2. A special case of Ramsey's theorem can be stated as follows: given any set of three people a, b and c, either two of them are friends with each other, or two of them are not friends with each other (assumming symmetry of the friendship relation). In order to show this, we can assume without loss of generality that if a and c are friends, then so are b and c: were this not the case, one can swap the roles of b and c, and obtain a new problem satisfying both the preconditions of the problem and our assumption. ■

Example 3. The pigeonhole problem asks whether m pigeons can be fit into n pigeonholes in such a way that no two pigeons are allocated to the same hole. To show this, one can assume without loss of generality that pigeons 1 and m are not simultaneously assigned to holes n and 1 simultaneously. If this happened, pigeon 1 can be moved to hole 1 while pigeon m is moved to hole n. This operation applied to any solution of the problem yields another solution of the problem, and additionally our assumption is fulfilled. ■

In either case, assumptions without loss of generality are not implied by the problem, but rather just consistent with it. The argument to introduce them is similar in both cases: if, under the conditions of the problem, the assumption does not hold, then one can perform a transformation that again falls into the conditions of the problem, and additionally the assumption holds true after the transformation.

4.1 Satisfiability preservation as a proof invariant

One pleasant and intuitive property of more traditional proof systems is truth preservation: if we can derive G from F, then we know that every model of F is a model of G. This property holds true for DRUP proofs, i.e. DPR proofs without PR introduction instructions, since both RUP introduction and clause deletion are truth-preserving operations. Truth equivalence cannot be concluded, for arbitrary clause deletion may allow additional models. Truth preservation acts as a semantic invariant: if we consider the set M of all interpretations satisfying F, then the accumulated formulas F' throughout the formula have the property that M is a set of satisfying interpretations for F', and in particular for G. Hence, one such DRUP proof acts as a witness that all models of F are models of G. Truth preservation is a very strong invariant. For example, effective methods to construct interpolants from resolution or DRUP proofs implicitly rely heavily on truth preservation [34], and reasoning about non-truth-preserving proofs is notably harder and often involves global reasoning about proofs [15, 32, 36].

Truth preservation is nevertheless lost once PR clause introduction is considered. Introducing a PR clause is only satisfiability-equivalent, but in general is not truth-preserving. Satisfiabilility-preservation in PR introduction, together with truth-preservation of deletions and RUP introductions, only establishes satisfiability preservation as a semantic invariant of PR proofs. In particular, the property preserved by the proof is that the set of interpretations satisfying each accumulated formula is nonempty. A reasonable question is then whether this is the strongest semantic invariant preserved by DPR proofs: stronger invariants would allow us to extract more information and have a more local understanding of proofs. In this regard, a converse to the satisfiability-preservation result appeared in [30]:

Theorem 3. *Let F and G be CNF formulas such that $F \vDash_{sat} G$. Then, there exists a DPR proof that derives G from F.*

This result seems at first discouraging. One interpretation of it is that the existence of a proof does not yield stronger semantic invariants than satisfiability preservation. Another interpretation is that no stronger semantic invariant can be obtained within propositional logic. As

we will see in the rest of this paper, moving our framework to the fringes of these understandings (in the former case, by considering the proof itself instead of its mere existence; in the latter, by understanding DPR inferences as acting over OCNFs) leads to new semantic invariants.

4.2 Proof-dependent semantic invariants

Our understanding of satisfiability preservation in DPR proofs boils down to [17, Theorem 1] which states that if F is a CNF formula and C is a PR clause in F upon a cube Q, then the existence of *some* model of F implies the existence of *some* model of $F \cup \{C\}$. This is actually an understatement: a careful look into the proof of the result from [17] shows that we know precisely how to transform *every* model of F into a model of $F \cup \{C\}$.

Theorem 4. *Let F be a CNF formula and C a PR clause in F upon some cube Q. Then, for every interpretation I satisfying F, the interpretation $I + \left(Q :- \overline{C}\right)$ satisfies $F \cup \{C\}$.*

Theorem 4 gives a new view on PR introduction, from which the relation to assumptions without loss of generality becomes apparent: if an interpretation satisfying the problem F violates the assumption C, then we refine the interpretation in such a way that the resulting interpretation satisfies both the problem and the assumption. In the case of PR introduction, interpretation refinement is only allowed in the form of block overwrites: specific literals, given by the witness cube Q are set to true or false. In particular, PR clauses cannot (directly) encode refinements where a variable takes the truth value of another variable.

Example 4. We can encode the problem from Example 2 into the following CNF formula F, where variable $x_{u,v}$ is satisfied if u and v are friends:

$$[+x_{a,b},\ +x_{b,c}] \quad [-x_{a,b},\ -x_{b,c}] \quad [+x_{a,b},\ +x_{a,c}] \quad [-x_{a,b},\ -x_{a,c}] \quad [+x_{a,c},\ +x_{b,c}] \quad [-x_{a,c},\ -x_{b,c}]$$

This is the problem from Example 4 in [13]. There, the symmetry breaker $[-x_{a,c},\ +x_{b,c}]$ is introduced which expresses the assumption without loss of generality given in Example 2. This clause can be introduced as a PR clause in F upon the witness cube $\langle -x_{a,c},\ +x_{b,c}\rangle$, and the latter expresses precisely the swap of the roles of a and b. ■

Example 5. The pigeonhole problem from Example 3 is expressed as the CNF formula F containing the following clauses:

$$\{[+x_{i,1} \ldots +x_{i,m}] \mid 1 \leq i \leq m\} \cup \{[-x_{i,k},\ -x_{j,k}] \mid 1 \leq i < j \leq n \text{ and } 1 \leq k \leq m\}$$

where the variable $x_{u,v}$ is satisfied whenever pigeon u is in hole v. Our assumption without loss of generality from Example 3 is written as the clause $[+x_{1,m},\ -x_{n,1}]$. This clause can be introduced in F as a PR clause upon the witness cube $\langle -x_{1,m},\ -x_{n,1},\ +x_{1,1},\ +x_{n,m}\rangle$. As in the previous example, this cube represents the transformation proposed there, which moves pigeon 1 to hole 1, and pigeon n to hole m. This PR introduction corresponds to the first clause introduced in the DPR proof for the pigeonhole problem with $n+1$ pigeons and n holes presented in [17]. ■

Semantics in terms of assumption without loss of generality can be expressed, as given by Theorem 4, using conditional overwrites. By collecting introduced overwrite rules, new semantic invariants can be developed to circunvent the limitations posed by Theorem 3. These semantics do not depend only on the existence of a proof, but also on the specific features of the proof,

specifically witness cubes. In addition to the accumulated CNF formula throughout a proof, we recursively define the *accumulated rule string* throughout a DPR proof π as follows:

$$\mathrm{ars}(\epsilon) = \epsilon \quad \mathrm{ars}(\mathbf{i:}\, C,\, \pi) = \mathrm{ars}(\mathbf{d:}\, C,\, \pi) = \mathrm{ars}(\pi) \quad \mathrm{ars}(\mathbf{i:}\, C \blacktriangleright Q,\, \pi) = \big(Q :\!-\, \overline{C}\big),\, \mathrm{ars}(\pi)$$

Intuitively, $\mathrm{ars}(\pi)$ collects the set of overwrite rules that would be applied by every PR introduction throughout the proof π in the same order as they occur. Theorem 4 can then straightforwardly be extended to work over whole DPR proofs, which gives a much stronger semantics than satisfiability preservation:

Corollary 2. *Let F be a CNF formula and π be a DPR derivation of a CNF formula G from F. Then, for every interpretation I such that $I \vDash F$, we have $I + ars(\pi) \vDash G$.*

4.3 DPR proofs as truth-preserving proofs on UOCNFs

An alternative way to understand DPR proofs is to embed the interpretation transformations induced by PR introductions in Theorem 4 directly into the formula syntax. Overwrite logics are especially appropriate for this: from Theorem 4 it becomes apparent that PR introduction naturally operates over UOCNF formulas.

Corollary 3. *Let F be a CNF formula and C a PR clause in F upon some cube Q. Then, $F \vDash \nabla\big(Q :\!-\, \overline{C}\big).\, F \cup \{C\}$.*

Observe a key feature of this perspective on PR introduction: when considered as an operation over UOCNF formulas, it is not only a satisfiability-preserving procedure, but also a *truth*-preserving one. Hence, the question of what are the semantics preserved by DPR proofs becomes trivial under the UOCNF perspective. This means that DPR proofs can be construed as (truth-preserving!) proofs over UOCNF formulas as follows:

DPR instruction	inference over CNFs	inference over UOCNFs
$\mathbf{i:}\, C$	$F \vDash \quad F \cup \{C\}$	$\nabla\vec{\varepsilon}.\, F \vDash \nabla\vec{\varepsilon}.\, F \cup \{C\}$
$\mathbf{i:}\, C \blacktriangleright Q$	$F \vDash_{\mathrm{sat}} F \cup \{C\}$	$\nabla\vec{\varepsilon}.\, F \vDash \nabla\vec{\varepsilon},\, \big(Q :\!-\, \overline{C}\big).\, F \cup \{C\}$
$\mathbf{d:}\, C$	$F \vDash \quad F \setminus \{C\}$	$\nabla\vec{\varepsilon}.\, F \vDash \nabla\vec{\varepsilon}.\, F \setminus \{C\}$

Intuitively, in the same way that the accumulated CNF formula throughout a DPR proof applies the clause introductions and deletions specified by the proof, the accumulated UOCNF formula additionally applies the overwrites associated to PR introductions. In particular, a DPR proof π deriving a CNF formula G from a CNF formula F can be understood as deriving from an UOCNF formula $\nabla\vec{\varepsilon}.\, F$ the UOCNF $\nabla\vec{\varepsilon},\, \mathrm{ars}(\pi).\, G$. The semantics of DPR proofs then become clear in the way that one would expect from a traditional proof system:

Corollary 4. *Let π be a DPR proof deriving a UOCNF formula Σ from a UOCNF formula Π, in the sense explained above. Then, $\Pi \vDash \Sigma$.*

5 New insights with overwrite logic

In Section 4.3, a perspective on DPR proofs as proofs over UOCNF formulas was explored. However, the answer is rather unsatisfactory: although now we have truth-preservation as an invariant, the usual properties of truth-preserving proofs are not attained. In particular, proofs

$$\frac{\nabla\vec{\varepsilon}.\, C \qquad \nabla\vec{\varepsilon}.\, D}{\nabla\vec{\varepsilon}.\, C \otimes D}\ \text{res} \quad \text{if the resolvent } C \otimes D \text{ exists}$$

$$\frac{\nabla\vec{\varepsilon}.\, C}{\nabla\vec{\varepsilon}.\, C \cup D}\ \text{sub} \quad \text{if } \overline{C} \not\models D$$

$$\frac{\nabla\vec{\varepsilon}.\, []}{[]}\ \text{bot}$$

$$\frac{\nabla\vec{\varepsilon}.\, D \qquad \nabla\vec{\varepsilon}.\, C \cup D|_B}{\nabla\vec{\varepsilon}\big(B :\!-\, \overline{C}\big).\, D}\ \text{ow-wlog} \quad \text{if } B \not\models D \text{ and } \overline{C} \not\models D|_B$$

$$\frac{\nabla\vec{\varepsilon}.\, D}{\nabla\vec{\varepsilon}\big(B :\!-\, \overline{C}\big).\, D}\ \text{ow-taut} \quad \text{if } B \models D \text{ or } \overline{C} \models D|_B$$

$$\frac{}{\nabla\vec{\varepsilon}\big(B :\!-\, \overline{C}\big).\, C}\ \text{ow-ax} \quad \text{if } B \models C$$

Figure 4: Proof rules for the overwrite resolution proof system

are still interference-based: the whole CNF part of the formula is involved in the inference criterion, which precludes tree-shaped proofs where inferences depend exclusively on the presence of some clauses, instead of their absence. The choice of UOCNF logic is to blame for this limitation, since clauses cannot be split from the formulas. However, understanding DPR proofs from a OCNF perspective eliminates this disadvantange.

Observe that, if a DPR proof can derive the UOCNF formula $\nabla\vec{\varepsilon}.\, G$ from a formula F, then we have $F \models \nabla\vec{\varepsilon}.\, G \equiv \{\nabla\vec{\varepsilon}.\, C \mid C \in G\}$, where the latter is a OCNF formula. In particular, every overwrite clause $\nabla\vec{\varepsilon}.\, C$ independently follows from F, *regardless of other clauses*. This suggests that interference can then be avoided. Our goal is to enrich a typical proof system over clauses with resolution and subsumption as proof rules into a proof system over overwrite clauses, and show that such a proof system can be used to simulate DPR as a tree-shaped proof system. The inferences of this *overwrite resolution* proof system are shown in Figure 4.

The rules ow-res and ow-sub are simply generalizations of the resolution and subsumption inference rules for clauses. Together, and following the characterization of RUPs as clauses derived by SSSR chains, allow the simulation of RUP inferences in an OCNF setting. The rule ow-bot is necessary to eliminate the string of overwrite rules to ultimately derive a contradiction. Finally, the role of PR clause introduction is performed by the ow-wlog, ow-taut and ow-ax rules. Perhaps counterintuitively, introducing *any* clause C under an overwrite rule $\big(Q :\!-\, \overline{C}\big)$ only requires that $Q \models C$, as shown by the ow-ax rule; this is necessary but insufficient condition for C to be a PR clause upon Q. In essence, this inference simply proves that an assumption without loss of generality holds after the associated model transformation. However, the troublesome part of reasoning without loss of generality usually is arguing that after the transformation the model still satisfies the original conditions, i.e. showing that $F \models \nabla\big(B :\!-\, \overline{C}\big).\, F$. The rules ow-wlog and ow-taut allow to simulate interference reasoning with OCNF formulas:

Proposition 2. *Let C be a PR clause in a CNF formula F upon a cube Q, and $D \in F$. Then, either of the following hold: $Q \models D$, or $\overline{C} \models D|_Q$, or $C \cup D|_Q$ is a RUP in F.*

Limitations of interference-based proof systems One furtive consequence of our framework is that some inferences can be derived with this framework but are cannot be obtained within DPR proofs, due to the notion of accumulated formula inherent to interference-based proofs. Let us first express the problem within an interference framework, and then we show how this limitation can be overcome with the use of the finer-grained reasoning of tree-shaped proofs over overwrite clauses. Given a CNF formula F and a clause C candidate to be PR in Q, the notion of PR clause forces us to derive every clause in $F|_Q$ from $F|_{\overline{C}}$ as a RUP. By Proposition 2, this is equivalent to deriving $C \cup D|_Q$ as a RUP in F for every clause $D \in F$ for which the former expression makes sense, i.e. $Q \not\models D$ and $\overline{C} \not\models D|_Q$ must hold.

It might be the case that for some $D \in F$, the clause $C \cup D|_Q$ is not a RUP in F, but we may know how to derive it by RUP once a lemma D' has been introduced. Under the interference framework, one would then need to derive $C \cup D'|_Q$ by RUP, which may need an additional lemma D'', and so on. This influence of the whole formula, which interference is named after, makes it very hard to reason about PR introduction. This limitation is eliminated in our OCNF framework: since there is no notion of accumulated formula, there is no need for $\nabla\big(B :\!- \overline{C}\big).\, D$ to be derived for *all* clauses $D \in F$, and so we are able to derive more redundant clauses.

Example 6. Consider a clause C and a cube Q such that $Q \vDash C$, and let us assume that $Q \setminus C \neq \emptyset$. We provide a formula F and a clause $D \in F$ such that C is neither a RUP clause nor a PR clause in F *nor* in $F \setminus \{D\}$. In other words, C cannot be derived by neither of the following proof fragments:

$$\textbf{i:}\ C \qquad \textbf{i:}\ C \blacktriangleright Q \qquad \textbf{d:}\ D,\ \textbf{i:}\ C \qquad \textbf{d:}\ D,\ \textbf{i:}\ C \blacktriangleright Q$$

Nevertheless, it can be derived within our framework. Let us choose literals $l \in Q \cap C$ and $k \in Q \setminus C$, and variables x, y not occurring in C or Q. Consider the formula F given by clauses:

$$[l,\ k,\ -u,\ -v] \qquad \big[\overline{l},\ \overline{k},\ -u,\ -v\big] \qquad \big[\overline{l},\ k,\ +u,\ +v\big] \qquad \big[\overline{k},\ -u\big]$$

Then, C is not a RUP in F nor in $F \setminus \{D\}$, since unit propagation on $F \cup \overline{C}$ does not reach a contradiction. Furthermore, C is not a PR in F upon Q: although $Q \vDash \big\{[l,\ k,\ -u,\ -v],\ \big[\overline{l},\ k,\ +u,\ +v\big]\big\}$, and $C \cup \big[\overline{l},\ k,\ +u,\ +v\big]|_Q = C \cup [-u,\ -v]$ reaches contradiction by unit propagation using clauses $\big[\overline{k},\ -u\big]$ and $[l,\ k,\ -u,\ -v]$, the clause $C \cup \big[\overline{k},\ -u\big]|_Q = C \cup \{-u\}$ cannot itself be derived as a RUP in F. Moreover, C is not a PR either in $F \setminus \big\{\big[\overline{k},\ -u\big]\big\}$ upon Q, since the removed clause was needed to reach a contradiction before. However, in our framework, we can use rules ow-taut and ow-wlog to derive the overwrite clauses:

$$\nabla\big(Q :\!- \overline{C}\big).\, [l,\ k,\ -u,\ -v] \qquad \nabla\big(Q :\!- \overline{C}\big).\, \big[\overline{l},\ \overline{k},\ -u,\ -v\big] \qquad \nabla\big(Q :\!- \overline{C}\big).\, \big[\overline{l},\ k,\ +u,\ +v\big]$$

■

6 Conclusion

We introduced overwrite logic, a new extension of propositional logic with a connective which allows us to capture the meta-level reasoning behind interference-based proof systems.

We analyzed the logic from the point of view of expressivity and complexity. We learnt that while often more succinct than standard propositional logic, the satisfiability problem for overwrite logic remains in NP and can be efficiently reduced to SAT.

We shed new light on DPR and DRAT proofs by proposing to understand the non-monotone PR, and its restriction RAT, introduction rule as formalisation of the proof technique of assumptions without loss of generality common in mathematics.

We then show that any DPR (DRAT) proof can be seen as a truth preserving proof in overwrite logic. This much stronger invariant than mere satisfiability preservation restores monotonicity: the clause deletion operation simply becomes a promise not to reuse the deleted clause later in the proof. This simplifies reasoning about proofs, opening possibilities for non-trivial processing of proofs such as interpolation.

This resolves the question of implicit semantics of DRAT proofs only partially answered in previous work [30] and extends to the more general DPR proof system as well. Moreover, we use this framework to identify an intrinsic limitation of interference proof systems and propose how it could be overcome in the more general setting using overwrite logic.

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Note to reviewers Proofs of our original results are included in this appendix. They are intended to provide support for our claims during the review process, but the main body of the paper should be enough to understand the main ideas. In case the paper gets accepted, we will upload these proofs to a public access repository and include a link in the camera-ready version.

A OCNF embeds UOCNF

Proposition 1. *$\nabla(B :- \psi)$ distributes with $\vee$, $\wedge$ and $\neg$. In particular, $\nabla\vec{\varepsilon}.F \equiv \{\nabla\vec{\varepsilon}.\, C \mid C \in F\}$.*

Proof. We show distributivity of $\nabla(B :- \psi)$ with $\wedge$; the cases for $\vee$ and $\neg$ are similar. In particular, we show that $\nabla(B :- \psi).\, \varphi_1 \wedge \varphi_2 \equiv \nabla(B :- \psi).\, \varphi_1 \wedge \nabla(B :- \psi).\, \varphi_2$.

Let I be an interpretation. We have that $I \vDash \nabla(B :- \psi).\varphi_1 \wedge \varphi_2$ if and only if $I + (B :- \psi) \vDash \varphi_1 \wedge \varphi_2$. This is equivalent to both $I + (B :- \psi) \vDash \varphi_i$ for $i \in \{1, 2\}$, or equivalently, $I \vDash \nabla(B :- \psi).\, \varphi_i$. Distributivity then follows.

To see that $\nabla\vec{\varepsilon}.\, F \equiv \{\nabla\vec{\varepsilon}.\, C \mid C \in F\}$, observe that $F \equiv \bigwedge_{C\in F} C$, because F is finite. By distributivity, we have $\nabla\vec{\varepsilon}.\, F \equiv \bigwedge_{C\in F} \nabla\vec{\varepsilon}.\, C$. □

B PL exponentially simulates OPL

Lemma 1. *Let A and B be cubes. Then, the set of literals $B \cup A \setminus \overline{B}$ is a cube, and $I + A + B = I + (B \cup A \setminus \overline{B})$ holds for every interpretation I.*

Proof. We first show that the set $X = B \cup A \setminus \overline{B}$ is a cube, i.e. it is complement-free. Assume it is not; then, a literal l exists such that both l and $\bar{l}$ are in X. Since both A and B are complement-free, it cannot be the case that l and $\bar{l}$ are both in B or both in $A \setminus \overline{B}$. Hence, we can assume without loss of generality that $l \in B$ and $\bar{l} \in A \setminus \overline{B}$. But this is a contradiction, because the former implies that $\bar{l} \in \overline{B}$. By reduction to absurd, X is a cube.

We now show the equality from the lemma for an arbitrary interpretation I. Let x be any propositional variable.

- If $+x \in B$, then $(I + A + B)(x) = 1 = (I + X)(x)$, and similarly if $-x \in B$, then $(I + A + B)(x) = 0 = (I + X)(x)$.
- If $x \notin \mathrm{Var}(B)$ and $+x \in A$, then $(I + A + B)(x) = (I + A)(x) = 1$. Furthermore, we have $+x \in A \setminus \overline{B}$, so $(I + X)(x) = 1$. Similarly, if $x \notin \mathrm{Var}(B)$ and $-x \in A$, then $(I + A + B)(x) = 0 = (I + X)(x)$.
- Otherwise, $x \notin \mathrm{Var}(B) \cup \mathrm{Var}(A)$, so we also have $x \notin \mathrm{Var}(X)$. Then, we obtain $(I + A + B)(x) = (I + A)(x) = I(x) = (I + X)(x)$. □

Theorem 1. *Let φ be an OPL formula and A be a cube. For every interpretation I, we have $I \vDash \varphi|_A$ if and only if $I + A \vDash \varphi$. In particular, the PL formula $\varphi|_\emptyset$ is truth-equivalent to φ.*

Proof. We show the theorem by induction on the structure of φ. The base cases for $\varphi = \bot$ and $\varphi = \top$ are trivial. We now show the base case when $\varphi = x$ is a propositional variable. If $+x \in A$, then $x|_A = \top$, so $I \vDash \varphi|_A$; furthermore, $I + A \vDash x = \varphi$, so the equivalence follows. The case when $-x \in A$ is analogous; we now show the case when $x \notin \mathrm{Var}(A)$. In this case we have $I \vDash x$ if and only if $I + A \vDash x$, and since $x|_A = x$, the equivalence holds. This shows the base case when φ is a propositional variable.

We now assume that the theorem holds for OPL formulas ψ and η. If $\varphi = \psi \wedge \eta$, then we have $\varphi|_A = \psi|_A \wedge \eta|_A$. Then, we have $I \vDash \varphi|_A$ if and only if $I \vDash \psi|_A$ and $I \vDash \eta|_A$. By induction hypothesis, this is equivalent to $I + A \vDash \psi$ and $I + A \vDash \eta$, which in turn holds if and only if $I + A \vDash \psi \wedge \eta = \varphi$.

The cases $\varphi = \psi \vee \eta$ and $\varphi = \neg\psi$ are shown in a similar fashion. We now show the theorem for $\varphi = \nabla(B :\!- \psi).\eta$, where B is a cube. The reduct is then $\varphi|_A = (\psi|_A \vee \eta|_A) \wedge (\neg\psi|_A \vee \eta|_{B \cup A \setminus \overline{B}})$. Let I be an interpretation satisfying $\varphi|_A$; we show that $I + A$ satisfies φ.

- If $I + A$ satisfies ψ, then by induction hypothesis we know that $I \vDash \psi|_A$, so we conclude that I satisfies $\eta|_{B \cup A \setminus \overline{B}}$. Then, $I + (B \cup A \setminus \overline{B}) \vDash \eta$. By Lemma 1 we then have that $I + A + B \vDash \eta$.
- If $I + A$ falsifies ψ, then similarly we conclude that I satisfies $\eta|_A$, so $I + A \vDash \eta$.

We conclude that $I + A + (B :\!- \psi) \vDash \eta$, or equivalently, $I + A \vDash \nabla(B :\!- \psi).\, \eta = \varphi$. We now show the converse. Let I be an interpretation such that $I + A$ satisfies φ.

- If $I + A$ satisfies ψ, then $I + A + B$ satisfies η. We thus conclude that I satisfies both $\psi|_A$ and $\eta|_{B \cup A \setminus \overline{B}}$, where we have used Lemma 1.
- If $I + A$ does not satisfy ψ, then $I + A$ satisfies η; then we know that I satisfies both $\neg\psi|_A$ and $\eta|_A$.

Hence, I satisfies $(\psi|_A \vee \eta|_A) \wedge (\neg\psi|_A \vee \eta|_{B \cup A \setminus \overline{B}}) = \varphi|_A$. This shows the case $\varphi = \nabla(B :\!- \psi).\eta$ and concludes the proof by induction. □

C UOCNF linearly simulates OPL

We now formalize the results from Section 3.2. Instead of relying on the intuition with circuits, we take here a more direct way using the Tseitin procedure and formula renaming to express the overwrite operation.

C.1 Node and transform variables

To construct the simulation, we will need a way to obtain "fresh" variables that are used nowhere else, which will then be assigned truth values through definitions. Each such variable may represent a subformula of the OPL formula φ to be simulated, or the truth value of a propositional variable after an overwrite. We will call the former node variables, and the latter trans variables. At the same time, we want to avoid unnecessary duplication of variables that represent the same subformula or overwritten truth value, so that the end result is as close to (sharing) circuits as possible. In this section we present a formalization of this idea.

We aim to construct variables $\mathsf{node}(\psi)$ and $\mathsf{trans}(x, (B :\!- \psi))$, where ψ is a OPL formula, x is a propositional variable and B is a cube; we want these variables to be "new" not only with respect to $\mathrm{Var}(\varphi)$, but also among themselves: two trans variables should be the same only if their OPL formula parameter is the same, and so on. A naïve way to get away with this would be to simply extend the signature of propositional variables; but observe that these variables may occur themselves in OPL formulas, which may in turn occur in the parameters of node and trans variables, an we would need to extend yet again the signature. Extending the signature infinitely many times leaves the question whether the set of variables is still countable, and it feels very counterintuitive that we need an uncountably infinite signature just to perform a Tseitin-like procedure. Instead, we follow a somewhat more complicated approach, where we

treat node and trans as mappings. The results obtained follow what one would expect from the notion of new variables, and we include only for the purpose of formality.

Lemma 2. *Let X be a countably infinite set, and $Y \subseteq X$ be a finite set. Then, there exists a partition $(X_i)_{i\in\mathbb{N}}$ of X such that X_i is countably infinite for each $i \in \mathbb{N}$, and furthermore $Y \subseteq X_0$.*

Proof. The set $\mathbb{N}^2$ is countably infinite, so we can find a bijection $\Phi\colon \mathbb{N}^2 \longrightarrow \mathbb{N}$. Consider sets $L_i = \{(a,b) \in \mathbb{N}^2 \mid a = i\}$ defined for each $i \in \mathbb{N}$. Then, $(L_i)_{i\in\mathbb{N}}$ forms a partition of $\mathbb{N}$. Now, X is countably infinite, so there is a bijection $\Psi\colon \mathbb{N} \longrightarrow X$. Define sets $X'_i = \Psi(\Phi(L_i))$ for each $i \in \mathbb{N}$. Since $\Psi\colon \Phi$ is a bijection, $(X'_i)_{i\in\mathbb{N}}$ forms a partition of X. Furthermore, since each of the L_i is countably infinite, then so are the X'_i.

The obtained partition does not necessarily satisfy $Y \subseteq X'_0$. However, since Y is finite, we can find a bijection $\sigma\colon X \longrightarrow X$ such that $\sigma(Y) \subseteq X'_0$. Then, we can define $X_i = \sigma^{-1}(X'_i)$ for all $i \in \mathbb{N}$, which fulfills our requirements. □

For the rest of this section, we denote by **Var** the set of all propositional variables, by **Wff** the set of all OPL formulas, and by **Vr** the set of pairs $(p, (B :\!- \psi))$ where p is a propositional variable and $(B :\!- \psi)$ is a (not necessarily cubic) overwrite rule. Now, let $(V_i)_{i\in\mathbb{N}}$ be any partition of **Var** into countably infinitely many sets, We define the following sets for all $i \in \mathbb{N}$:

$$\begin{aligned} V_{\leq i} &= \cup_{j=0}^{i} V_i \\ \mathbf{Wff}_{\leq i} &= \{\psi \in \mathbf{Wff} \mid \mathrm{Var}(\varphi) \subseteq V_{\leq i}\} \\ \mathbf{Vr}_{\leq i} &= \{(p, (B :\!- \psi)) \mid \{p\} \cup \mathrm{Var}(B) \cup \mathrm{Var}(\psi) \subseteq V_{\leq i}\} \\ \mathbf{Wff}_i &= \mathbf{Wff}_{\leq i} \setminus \mathbf{Wff}_{\leq i-1} \\ \mathbf{Vr}_i &= \mathbf{Vr}_{\leq i} \setminus \mathbf{Vr}_{\leq i-1} \end{aligned}$$

where we tacitly assume $\mathbf{Wff}_{\leq -1} = \mathbf{Vr}_{\leq -1} = \emptyset$.

Proposition 3. *Let V be a finite set of propositional variables. Then, there exists a partition $(V_i)_{i\in\mathbb{N}}$ of* **Var***, and mappings*

$$\mathit{node}\colon \mathbf{Wff} \longrightarrow \mathbf{Var} \qquad\qquad \mathit{trans}\colon \mathbf{Vr} \longrightarrow \mathbf{Var}$$

such that the following hold:

1. *The mappings* node *and* trans *are injective.*
2. *The set of variables V is contained in V_0.*
3. *For all $i \in \mathbb{N}$, the image sets $\mathit{node}(\mathbf{Wff}_i)$ and $\mathit{trans}(\mathbf{Vr}_i)$ are contained in V_{i+1}.*
4. *The image sets $\mathit{node}(\mathbf{Wff})$ and the image set $\mathit{trans}(\mathbf{Vr})$ are disjoint.*

Proof. Since V is finite, we can find two countably infinite subsets V^{node} and V^{trans} of **Var** such that the three sets V, V_{node} and V_{trans} form a partition of **Var**. By Lemma 2, we can find partitions $(V_i^{\mathsf{node}})_{i\in\mathbb{N}}$ and $(V_i^{\mathsf{trans}})_{i\in\mathbb{N}}$ of $V \cup V^{\mathsf{node}}$ and $V \cup V^{\mathsf{trans}}$ respectively, such that each V_i^{node} and V_i^{trans} is countably infinite for $i \in \mathbb{N}$, and furthermore $V \subseteq V_0^{\mathsf{node}}$ and $V \subseteq V_0^{\mathsf{trans}}$. We choose $V_i = V_i^{\mathsf{node}} \cup V_i^{\mathsf{trans}}$ for each $i \in \mathbb{N}$; property 2 follows straightforward.

We now recursively construct mappings:

$$\mathsf{node}_i\colon \mathbf{Wff}_{\leq i-1} \longrightarrow V_{\leq i} \qquad\qquad \mathsf{trans}_{i-1}\colon \mathbf{Vr}_{\leq i-1} \longrightarrow V_{\leq i}$$

for each $i \in \mathbb{N}$ such that properties 1, 3, and 4 are satisfied where applicable, and furthermore $\mathsf{node}_j \subseteq \mathsf{node}_i$ and $\mathsf{trans}_j \subseteq \mathsf{trans}_i$ for $j \in \{0, \dots, i\}$. The base case $i = 0$ is trivial, since $\mathbf{Wff}_{\leq i-1} = \mathbf{Vr}_{\leq i-1} = \emptyset$, and so taking node_j and trans_j as the empty functions trivially satisfies the conditions.

Now assume that functions $\mathsf{node}_0, \dots, \mathsf{node}_i, \mathsf{trans}_0, \dots, \mathsf{trans}_i$ satisfying the corresponding conditions have been defined. The sets $\mathbf{Wff}_i$ and $\mathbf{Vr}_i$ are countable, and the sets V_{i+1}^{node} and V_{i+1}^{trans} are countably infinite. Hence, there exist injective mappings

$$N \colon \mathbf{Wff}_i \longrightarrow V_{i+1}^{\mathsf{node}} \qquad\qquad T \colon \mathbf{Vr}_i \longrightarrow V_{i+1}^{\mathsf{trans}}$$

Observe that $\mathbf{Wff}_i$ and $\mathbf{Wff}_{\leq i-1}$ form a partition of $\mathbf{Wff}_{\leq i}$, and similarly $\mathbf{Vr}_i$ and $\mathbf{Vr}_{\leq i-1}$ form a partition of $\mathbf{Vr}_{\leq i-1}$. Furthermore, both V_{i+1}^{node} and V_{i+1}^{trans} are included in $V_{\leq i+1}$. Then, we can define mappings as follows:

$$\begin{aligned} \mathsf{node}_{i+1} &\colon \mathbf{Wff}_{\leq i} \longrightarrow V_{\leq i+1} & \mathsf{node}_{i+1} &= \mathsf{node}_i \cup N \\ \mathsf{trans}_{i+1} &\colon \mathbf{Vr}_{\leq i} \longrightarrow V_{\leq i+1} & \mathsf{trans}_{i+1} &= \mathsf{trans}_i \cup T \end{aligned}$$

The property that $\mathsf{node}_j \subseteq \mathsf{node}_{i+1}$ and $\mathsf{trans}_j \subseteq \mathsf{trans}_{i+1}$ for all $j \in \{0, \dots, i+1\}$ is then straightforward, and given the codomains of N and T, property 3 follows as well. Since the images of node_i, trans_i, N and T are pairwise disjoint, and all four mappings are injective on their own, then properties 1 and 4 also hold true.

By induction, mappings node_i and trans_i satisfying the conditions above exist. Now, define mappings

$$\mathsf{node} = \bigcup_{i \in \mathbb{N}} \mathsf{node}_i \qquad\qquad \mathsf{trans} = \bigcup_{i \in \mathbb{N}} \mathsf{trans}_i$$

which exist because of the condition $\mathsf{node}_i \subseteq \mathsf{node}_j$ and $\mathsf{trans}_i \subseteq \mathsf{trans}_j$ for all $i \leq j$. The mappings node and trans then satisfy the requirements of the claim. □

In the following, we explain on how to translate a fixed OPL formula φ into a UOCNF formula. By Proposition 3, we choose a partition $(V_i)_{i \in \mathbb{N}}$ and mappings node and trans with the requirements specified there for $V = \mathrm{Var}(\varphi)$.

C.2 Breaking an OPL formula into definitions

In this section we present a function Defs which transforms an OPL formula into the definitions that correspond to its circuit in Section 3.2. The translation of the overwrite connective, as in the circuit presented there, requires renaming some variables in a subformula, so we also define a renaming function Rn which takes care of that.

Let us first define the renaming function, since Defs will be defined in terms of it. Given a (not necessarily cubic) overwrite rule $(B :\!- \psi)$, where B is a cube and ψ is an arbitrary OPL formula, we define the *renaming* of an OPL formula by $(B :\!- \psi)$ acting as follows on propositional variables:

$$\mathrm{Rn}_{(B :\!- \psi)}(p) = \begin{cases} p & \text{if } p \notin \mathrm{Var}(B) \\ \mathsf{trans}(p, (B :\!- \psi)) & \text{if } p \in \mathrm{Var}(B) \end{cases}$$

and we extend it homomorphically to all other expressions, i.e.:

$$\begin{aligned}
\mathrm{Rn}_{(B:-\psi)}(\neg\eta) &= \neg\mathrm{Rn}_{(B:-\psi)}(\eta)\\
\mathrm{Rn}_{(B:-\psi)}(\eta \vee \chi) &= \mathrm{Rn}_{(B:-\psi)}(\eta) \vee \mathrm{Rn}_{(B:-\psi)}(\chi)\\
\mathrm{Rn}_{(B:-\psi)}(\eta \wedge \chi) &= \mathrm{Rn}_{(B:-\psi)}(\eta) \wedge \mathrm{Rn}_{(B:-\psi)}(\chi)\\
\mathrm{Rn}_{(B:-\psi)}(+p) &= +\,\mathrm{Rn}_{(B:-\psi)}(p)\\
\mathrm{Rn}_{(B:-\psi)}(-p) &= -\,\mathrm{Rn}_{(B:-\psi)}(p)\\
\mathrm{Rn}_{(B:-\psi)}(Q) &= \big\{\mathrm{Rn}_{(B:-\psi)}(l) \mid l \in Q\big\}\\
\mathrm{Rn}_{(B:-\psi)}(\nabla(A :- \chi).\,\eta) &= \nabla\big(\mathrm{Rn}_{(B:-\psi)}(A) :- \mathrm{Rn}_{(B:-\psi)}(\chi)\big).\,\mathrm{Rn}_{(B:-\psi)}(\eta)
\end{aligned}$$

Intuitively, $\mathrm{Rn}_{(B:-\psi)}(\eta)$ provides a new formula where all the variables whose truth value would be modified by the rule $(B :- \psi)$ have been replaced by new variables parameterized by the rule itself and the original variable.

Now we can define the set of definitions associated to an OPL formula, following the circuit-based approach from Section 3.2. A definition is a PL formula of the form $x \leftrightarrow \psi$ where x is a propositional variable and ψ is a PL formula. The following recursive function describes a set of PL formulas for each OPL formula:

$$\begin{aligned}
\mathrm{Defs}(\top) &= \{\mathsf{node}(\top) \leftrightarrow \top\}\\
\mathrm{Defs}(\bot) &= \{\mathsf{node}(\top) \leftrightarrow \bot\}\\
\mathrm{Defs}(p) &= \{\mathsf{node}(p) \leftrightarrow p\}\\
\mathrm{Defs}(\neg\psi) &= \{\mathsf{node}(\neg\psi) \leftrightarrow \neg\mathsf{node}(\psi)\} \cup \mathrm{Defs}(\psi)\\
\mathrm{Defs}(\psi \vee \eta) &= \{\mathsf{node}(\psi \vee \eta) \leftrightarrow \mathsf{node}(\psi) \vee \mathsf{node}(\eta)\} \cup \mathrm{Defs}(\psi) \cup \mathrm{Defs}(\eta)\\
\mathrm{Defs}(\psi \wedge \eta) &= \{\mathsf{node}(\psi \wedge \eta) \leftrightarrow \mathsf{node}(\psi) \wedge \mathsf{node}(\eta)\} \cup \mathrm{Defs}(\psi) \cup \mathrm{Defs}(\eta)\\
\mathrm{Defs}(\nabla(B :- \psi).\,\eta) &= \big\{\mathsf{node}(\nabla(B :- \psi).\,\eta) \leftrightarrow \mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\eta)\big)\big\} \cup\\
&\quad \{\mathsf{trans}(p, (B :- \psi)) \leftrightarrow p \vee \mathsf{node}(\psi) \mid +p \in B\} \cup\\
&\quad \{\mathsf{trans}(p, (B :- \psi)) \leftrightarrow p \wedge \neg\mathsf{node}(\psi) \mid -p \in B\} \cup\\
&\quad \mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\eta)\big) \cup \mathrm{Defs}(\psi)
\end{aligned}$$

The definitions in $\mathrm{Defs}(\eta)$ relate node variables to the truth value of subformulas (or renamed subformulas) of η in terms of its immediate subformulas, and trans variables to the truth value of variables after a conditional overwrite in terms of the overwrite rule.

Throughout the rest of this section, we show that the dependencies between variables determined by biimplications in $\mathrm{Defs}(\eta)$ define a directed acyclic graph, where the only independent nodes are the variables of η. First of all, we must ensure that the $\mathrm{Defs}(\eta)$ is well-defined (i.e. terminating) and yields a set of definitions. It turns out, the latter can only be guaranteed if all the variables in $\mathrm{Var}(\eta)$ are in the same variable set V_i, since we need to ensure that node and trans variables are distinct from each other in a stratified way.

Proposition 4. *Let η be an OPL formula. Then, for each recursive call $\mathrm{Defs}(\eta')$ immediately spawned from $\mathrm{Defs}(\eta)$, we have $|\eta'| < |\eta|$. In particular, Defs is terminating. Furthermore, $\mathrm{Defs}(\eta)$ is a finite set of biimplications containing at most $|\eta|$ elements.*

Proof. The first claim follows by inspection, once one considers that $\big|\mathrm{Rn}_{(B:-\psi)}(\eta')\big| = |\eta'|$. The second claim follows by induction on the size of η; we show here a few cases, and the rest follow analogously.

- If p is a propositional variable, then $\#\mathrm{Defs}(p) = 1 = |p|$.
- If ψ and η are OPL formulas, then

$$\#\mathrm{Defs}(\psi \vee \eta) \leq 1 + \#\mathrm{Defs}(\psi) + \#\mathrm{Defs}(\eta) \leq 1 + |\psi| + |\eta| \leq |\psi \vee \eta|$$

- If B is a cube and ψ and η are OPL formulas, then

$$\begin{aligned}\#\mathrm{Defs}(\nabla(B :\!- \psi).\, \eta) &\leq 1 + \#B + \#\mathrm{Defs}(\psi) + \#\mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\eta)\big) \leq \\ &\leq 1 + |B| + |\psi| + |\eta| \leq |\nabla(B :\!- \psi).\, \eta|\end{aligned}$$

□

Lemma 3. *Let η be an OPL formula, and $(B :\!- \psi)$ be an overwrite rule. Define the following formulas*

$$\Delta^+ = \bigwedge_{+p \in B} (p \vee \mathit{node}(\psi)) \qquad\qquad \Delta^- = \bigwedge_{-p \in B} (p \wedge \neg \mathit{node}(\psi))$$

Then, $\Delta^+ \wedge \Delta^- \wedge (\psi \leftrightarrow \mathit{node}(\psi)) \vDash \eta \leftrightarrow \mathit{Rn}_{(B:-\psi)}(\eta)$.

Proof. By a rutinary induction on the structure of η. □

Proposition 5. *Let η be an OPL formula. Then, we have*

$$\bigwedge_{x \leftrightarrow \delta \in \mathit{Defs}(\eta)} (x \leftrightarrow \delta) \vDash \eta \leftrightarrow \mathit{node}(\eta)$$

Proof. By induction on $|\eta|$. The base case $|\eta| = 0$ is trivial, because there is no formula of size 0. We now show the induction case. Assume that the claim holds for all subformulas of φ of size less or equal than some k, and we show the claim for a subformula η of size $k+1$. We only show some cases; the rest follow similarly.

- If η is $\top$, $\bot$ or a propositional variable, then the claim is trivial.
- If $\eta = \psi \vee \chi$, then we must prove

$$\begin{aligned}(\mathsf{node}(\psi \vee \chi) \leftrightarrow \mathsf{node}(\psi) \vee \mathsf{node}(\chi)) \wedge \\ \bigwedge_{x \leftrightarrow \delta \in \mathrm{Defs}(\psi)} (x \leftrightarrow \delta) \wedge \\ \bigwedge_{x \leftrightarrow \delta \in \mathrm{Defs}(\chi)} (x \leftrightarrow \delta) \vDash (\psi \vee \chi) \leftrightarrow \mathsf{node}(\psi \vee \chi)\end{aligned}$$

  Let I be an interpretation satisfying the left-hand side of this implication. Since χ and ψ are subformulas of η with strictly smaller size, we know by induction hypothesis:

$$\bigwedge_{x \leftrightarrow \delta \in \mathrm{Defs}(\psi)} (x \leftrightarrow \delta) \vDash \psi \leftrightarrow \mathsf{node}(\psi)$$

$$\bigwedge_{x \leftrightarrow \delta \in \mathrm{Defs}(\chi)} (x \leftrightarrow \delta) \vDash \chi \leftrightarrow \mathsf{node}(\chi)$$

  Thus, I satisfies $\psi \vee \chi$ if and only if I satisfies either $\mathsf{node}(\psi)$ or $\mathsf{node}(\chi)$, and since I satisfies the biimplication $\mathsf{node}(\psi \vee \chi) \leftrightarrow \mathsf{node}(\psi) \vee \mathsf{node}(\chi)$, this holds if and only if I satisfies $\mathsf{node}(\psi \vee \chi)$.

- If $\eta = \nabla(B :\!- \psi).\, \chi$, then we must prove

$$\begin{aligned}\big(\mathsf{node}(\nabla(B :\!- \psi).\, \chi) \leftrightarrow \mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)\big) \wedge \\ \bigwedge_{+p\in B} (\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \vee \mathsf{node}(\psi)) \wedge \\ \bigwedge_{-p\in B} (\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \wedge \neg\mathsf{node}(\psi)) \wedge \\ \bigwedge_{x\leftrightarrow\delta\in\mathrm{Defs}(\psi)} (x \leftrightarrow \delta) \wedge \\ \bigwedge_{x\leftrightarrow\delta\in\mathrm{Defs}\left(\mathrm{Rn}_{(B:-\psi)}(\chi)\right)} (x \leftrightarrow \delta) \vDash (\nabla(B :\!- \psi).\, \chi) \leftrightarrow \mathsf{node}(\nabla(B :\!- \psi).\, \chi)\end{aligned}$$

  Let I be an interpretation satisfying the left-hand side. Let us first assume I satisfies $\nabla(B :\!- \psi).\, \chi$. Then, $I + (B :\!- \psi)$ satisfies χ. Observe that $I + (B :\!- \psi)$ assigns to p the same truth value as I assigns to $\mathsf{trans}(p, (B :\!- \psi))$ for every $p \in \mathrm{Var}(B)$. Since I satisfies the conditions of Lemma 3, we can apply it to conclude that I satisfies $\nabla(B :\!- \psi).\, \chi$ if and only if I satisfies $\mathrm{Rn}_{(B:-\psi)}(\chi)$, and then since the size of ψ is smaller than that of η, we can derive from the induction hypothesis that I satisfies η if and only if I satisfies $\mathsf{node}(\eta)$.

□

C.3 Definitions specify a circuit

Given a set of definitions, one may consider the dependency relation it induces. In particular, if we have definitions $x_i \leftrightarrow \delta_i$, we can define a graph where vertices are given by the defined variables x_i, and a directed edge between x_i and x_j exists whenever $x_i \in \mathrm{Var}(\delta_j)$, i.e. when x_i depends on x_j. In this setting, it is important to ensure that the resulting graph is acyclic, since otherwise the definitions would not make much sense. Further complications arise from the possibility that multiple definitions exist for the same variable, and any reasonable setting requires the presence of independent variables that decide the truth value of the defined variables. We now proceed to formalize this setting, and show that $\mathrm{Defs}(\varphi)$ satisfies that formalization.

Consider a set D of definitions. We say that D is *univalent* whenever, given two definitions $x \leftrightarrow \delta$ and $y \leftrightarrow \gamma$ from D, we have $\delta = \gamma$ whenever $x = y$. The set D is *acyclic* whenever, for every sequence $(x_i \leftrightarrow \delta_i)_{i=0}^{n}$ of definitions in D with $x_{i-1} \in \delta_i$ for $1 \leq i \leq n$, we have $x_n \notin \delta_0$. Given a set V of variables, we say that D is *determined* by V whenever, for all definitions $x \leftrightarrow \delta$ in D and every variable $y \in \mathrm{Var}(\delta) \setminus V$, there is a definition of the form $y \leftrightarrow \gamma$ in D; and we say that D *respects* V whenever for every definition $x \leftrightarrow \delta$ in D we have $x \notin V$. A *definition circuit* over V is a set of definitions which is univalent, acyclic, and is determined by and respects V.

We will later show that finite definition circuits are interesting because their definitions can be enumerated in an order compatible with the dependency relation, similarly to finding a topological ordering on a logical circuit. We first show that definition sets generated by Defs are univalent. In fact, every definition *ever* generated by Defs depends exclusively on the defined variable.

Lemma 4. *Let η be any OPL formula. Let $x \leftrightarrow \delta$ be a definition in $\mathrm{Defs}(\eta)$. Then, the following hold:*

- *Either $x = \mathsf{node}(\psi)$ or $x = \mathsf{trans}(p, (B :\!- \psi))$, for some OPL formula ψ, some propositional variable p and some cube B.*

- *If* $x = \mathsf{node}(\top)$*, then* $\delta = \top$*.*
- *If* $x = \mathsf{node}(\bot)$*, then* $\delta = \bot$*.*
- *If* $x = \mathsf{node}(p)$ *for a propositional variable* p*, then* $\delta = p$*.*
- *If* $x = \mathsf{node}(\neg\psi)$ *for an OPL formula* ψ*, then* $\delta = \neg\mathsf{node}(\psi)$*.*
- *If* $x = \mathsf{node}(\psi \vee \chi)$ *for OPL formulas* ψ *and* χ*, then* $\delta = \mathsf{node}(\psi) \vee \mathsf{node}(\chi)$*.*
- *If* $x = \mathsf{node}(\psi \wedge \chi)$ *for OPL formulas* ψ *and* χ*, then* $\delta = \mathsf{node}(\psi) \wedge \mathsf{node}(\chi)$*.*
- *If* $x = \mathsf{node}(\nabla(B :\!- \psi).\, \chi)$ *for OPL formulas* ψ *and* χ*, and a block* B*, then* $\delta = \mathsf{node}\big(Rn_{(B:-\psi)}(\chi)\big)$*.*
- *If* $x = \mathsf{trans}(p, (B :\!- \psi))$ *for a propositional variable* p *and a rule* $(B :\!- \psi)$*, then* $p \in \mathrm{Var}(B)$*. Furthermore, if* $+p \in B$*, then* $\delta = p \vee \mathsf{node}(\psi)$*; and if* $-p \in B$*, then* $\delta = p \wedge \neg\mathsf{node}(\psi)$*.*

Proof. By a very uninteresting induction on the size of η. □

Proposition 6. *Let* $x \leftrightarrow \delta$ *and* $x \leftrightarrow \delta'$ *be definitions in* $\mathit{Defs}(\eta)$ *and* $\mathit{Defs}(\eta')$ *respectively. Then,* $\delta = \delta'$*.*

Proof. By Lemma 4, we know that either $x = \mathsf{node}(\psi)$ or $x = \mathsf{trans}(p, (B :\!- \psi))$, for some OPL formula ψ, some propositional variable p and some cube B. A case-by-case analysis shows that Lemma 4 forces $\delta = \delta'$. □

We now proceed to show that the definition sets generated by Defs are acyclic. To do so, we find measures in the definitions that are modified in irreversible ways by the dependency relation induced by definitions.

Lemma 5. *Let* η *be an OPL formula. Let* $x \leftrightarrow \delta$ *be a definition in* $\mathit{Defs}(\eta)$*. Then, either of the following holds:*

- $x = \mathsf{node}(\eta')$ *for some OPL formula* η'*, and for every variable* $y \in \mathrm{Var}(\delta)$ *we have* $y = \mathsf{node}(\eta'')$ *for another OPL formula with* $|\eta'| > |\eta''|$*.*
- $x = \mathsf{trans}(p, (B :\!- \eta'))$ *for some propositional variable* p*, some cube* B*, and some OPL formula* η*, and for every variable* $y \in \mathrm{Var}(\delta)$*, we have* $y = p$ *or* $y = \mathsf{node}(\eta'')$ *for some OPL formula* η''*.*

Proof. By inspection. □

Proposition 7. *Let* η *be an OPL formula, and consider a finite sequence* $(x_i \leftrightarrow \delta_i)_{i=0,\ldots,n}$ *of definitions in* $\mathit{Defs}(\eta)$ *such that, for all* $1 \leq i \leq n$ *we find* $x_{i-1} \in \mathrm{Var}(\delta_i)$*. Then,* $x_n \notin \mathrm{Var}(\delta_0)$*.*

Proof. Let us assume such a finite sequence of definitions exists with $x_n \in \mathrm{Var}(\delta_0)$ and reason by contradiction. We know from Lemma 4 that all the x_i are either images of node or images of trans.

- If there is some definition $x_k \leftrightarrow \delta_k$ with $x_k = \mathsf{node}(\eta_k)$ for some OPL formula η_k, then we can apply iteratively Lemma 5 to deduce that all $x_i = \mathsf{node}(\eta_i)$ for all $0 \leq i \leq n$, where $\eta_0, \ldots, \eta_n$ are OPL formulas. Furthermore, that lemma tells us that we would then have

$$|\eta_i| < |\eta_{i-1}| < \cdots < |\eta_0| < |\eta_n| < \cdots < |\eta_{i+1}| < |\eta_i|$$

  which is a contradiction.

- Otherwise, we have $x_i = \mathsf{trans}(p_i, (B_i :\!- \psi_i))$ for all $0 \leq i \leq n$. Since the images of trans and node are disjoint, Lemma 5 says that $x_{i-1} = p_i$ for all $1 \leq i \leq n$ and $x_n = p_0$. Now, we know there is a unique $m_0 \in \mathbb{N}$ such that $p_0 = x_n \in V_{m_0}$, so we have that the pair $(p_0, (B :\!- \psi_0))$ is in $\mathbf{Vr}_i$ for some $i \geq m_0$. This means that $\mathsf{trans}(p_i, (B_i :\!- \psi_i))$ is a propositional variable in V_{m_1} where $m_1 = i+1 > m_0$. Iterating this argument, we conclude that each p_i is in some V_{m_i} with $m_0 < m_1 < \cdots < m_n$. Yet another application of the argument shows that p_0 is in V_k for some $k > m_n > m_0$. But $p_0 \in V_{m_0}$, and the V_j are pairwise disjoint, which means that $m_0 = k$. Hence, $m_0 < m_0$, which is a contradiction.

□

We now show that any set of definitions generated by Defs is determined by the variables occurring in the input formula. The proof goes as expected by induction.

Proposition 8. *Let η be an OPL formula. Then, $\mathrm{Defs}(\eta)$ is determined by $\mathrm{Var}(\eta)$.*

Proof. We show by induction on the size of η that the claim holds, and additionally there is a definition of the form $\mathsf{node}(\eta) \leftrightarrow \delta$ in $\mathrm{Defs}(\eta)$. We only show some cases, and the others can be shown analogously.

- If $\eta = \top$, then $\mathrm{Defs}(\eta) = \{\mathsf{node}(\top) \leftrightarrow \top\}$, and since $\mathrm{Var}(\top) = \emptyset$ the claim is trivial.
- If $\eta = p$ for some propositional variable p, then we have $\{\mathsf{node}(p) \leftrightarrow p\}$, and so the only variable in $\mathrm{Var}(p)$ is p, which is in $\mathrm{Var}(\eta)$, so the claim is trivial.
- If $\eta = \psi \vee \chi$, then:

$$\mathrm{Defs}(\psi \vee \chi) = \{\mathsf{node}(\psi \vee \chi) \leftrightarrow \mathsf{node}(\psi) \vee \mathsf{node}(\chi)\} \cup \mathrm{Defs}(\psi) \cup \mathrm{Defs}(\chi)$$

  Trivially, there is a definition of the form $\mathsf{node}(\psi \vee \chi) \leftrightarrow \delta$ in $\mathrm{Defs}(\psi \vee \chi)$. Let $x \leftrightarrow \delta'$ be a definition in $\mathrm{Defs}(\psi \vee \chi)$. If this definition is in $\mathrm{Defs}(\psi)$ or $\mathrm{Defs}(\chi)$, then the claim follows from the induction hypothesis considering that $\mathrm{Var}(\psi) \cup \mathrm{Var}(\chi) \subseteq \mathrm{Var}(\psi \vee \chi)$. Otherwise, we have $x = \mathsf{node}(\psi \vee \chi)$ and $\delta' = \mathsf{node}(\psi) \vee \mathsf{node}(\chi)$. Hence, $\mathrm{Var}(\delta') = \{\mathsf{node}(\psi), \mathsf{node}(\chi)\}$, and by induction hypothesis we know that there is a definition of the form $\mathsf{node}(\psi) \leftrightarrow \delta_\psi$ in $\mathrm{Defs}(\psi)$, and a definition of the form $\mathsf{node}(\chi) \leftrightarrow \delta_\chi$ in $\mathrm{Defs}(\chi)$, so the claim holds true.
- If $\eta = \nabla(B :\!- \psi).\, \chi$, then:

$$\begin{aligned}\mathrm{Defs}(\nabla(B :\!- \psi).\, \chi) = {} & \big\{\mathsf{node}(\nabla(B :\!- \psi).\, \chi) \leftrightarrow \mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)\big\} \cup \\ & \{\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \vee \mathsf{node}(\psi) \mid +p \in B\} \cup \\ & \{\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \wedge \neg\mathsf{node}(\psi) \mid -p \in B\} \cup \\ & \mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big) \cup \mathrm{Defs}(\psi)\end{aligned}$$

  Trivially, there is a definition of the form $\mathsf{node}(\nabla(B :\!- \psi).\, \chi) \leftrightarrow \delta$ in $\mathrm{Defs}(\eta)$. Now let $x \leftrightarrow \delta'$ be a definition in $\mathrm{Defs}(\psi \vee \chi)$.
  - If $x \leftrightarrow \delta'$ is $\mathsf{node}(\nabla(B :\!- \psi).\, \chi) \leftrightarrow \mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$, then the only variable in δ' is $\mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$, and by induction hypothesis there is a definition of the form $\mathsf{node}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big) \leftrightarrow \delta''$ in $\mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$.
  - If $x \leftrightarrow \delta'$ is $\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \vee \mathsf{node}(\psi)$ for some variable p with $+p \in B$, then the only variables in the right-hand side are p and $\mathsf{node}(\psi)$. Now, since $+p \in B$, we know that $p \in \mathrm{Var}(\eta)$. Furthermore, by induction hypothesis there is a definition of the form $\mathsf{node}(\psi) \leftrightarrow \delta''$ in $\mathrm{Defs}(\psi)$.

- The case when $x \leftrightarrow \delta'$ is $\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow p \wedge \neg\mathsf{node}(\psi)$ can be shown similarly to the previous case.
- If $x \leftrightarrow \delta'$ is in $\mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$, then observe that $\big|\mathrm{Rn}_{(B:-\psi)}(\chi)\big| = |\chi| < |\eta|$, so the claim holds for $\mathrm{Rn}_{(B:-\psi)}(\chi)$. Hence, for every variable $y \in \mathrm{Var}(\delta')$ we either have a definition $y \leftrightarrow \delta''$ in $\mathrm{Defs}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$, in which case the claim holds; or we have $y \in \mathrm{Var}\big(\mathrm{Rn}_{(B:-\psi)}(\chi)\big)$. Now, a very simple induction shows that such a variable can only be either in $\mathrm{Var}(\chi)$, or be a variable $y = \mathsf{trans}(p, (B :\!- \psi))$ for some $p \in \mathrm{Var}(B)$. The former case satisfies the claim because $\mathrm{Var}(\chi) \subseteq \mathrm{Var}(\eta)$; the latter case satisfies the claim because we have a definition of the form $\mathsf{trans}(p, (B :\!- \psi)) \leftrightarrow \delta''$ for every $p \in \mathrm{Var}(B)$.
- If $x \leftrightarrow \delta'$ is in $\mathrm{Defs}(\psi)$, then the claim holds by induction hypothesis.

□

Finally, we would like to show that every set of definitions $\mathrm{Defs}(\eta)$ respects the set of variables of η. However, in general this claim does not hold; one needs to restrict $\mathrm{Defs}(\eta)$ not to include any node or trans variable.

Lemma 6. *Let η be an OPL formula. Then, for every definition $x \leftrightarrow \delta$ in $\mathrm{Defs}(\eta)$, the variable x is either in the image set of node or in the image set of trans.*

Proof. By a very uninteresting induction. □

Proposition 9. *Let η be an OPL formula such that $\mathrm{Var}(\eta) \subseteq V_0$. Then, the set of definitions $\mathrm{Defs}(\eta)$ respects $\mathrm{Var}(\eta)$.*

Proof. Let $x \leftrightarrow \delta$ be a definition in $\mathrm{Defs}(\eta)$. Then, by Lemma 6, either of the following holds:

- $x = \mathsf{node}(\chi)$ for some OPL formula χ. Then, there is exactly one $i \in \mathbb{N}$ such that $\chi \in \mathbf{Wff}_i$, and then by Proposition 3 we have $\mathsf{node}(\chi) \in V_{i+1}$, which is disjoint with V_0. Therefore, $x \notin \mathrm{Var}(\eta)$.
- $x = \mathsf{trans}(p, (B :\!- \psi))$ for some propositional variable p and some overwrite rule $(B :\!- \psi)$. Then, there is exactly one $i \in \mathbb{N}$ such that the pair $(p, (B :\!- \psi))$ is in $\mathbf{Vr}_i$. Then, by Proposition 3 we have $\mathsf{trans}(p, (B :\!- \psi)) \in V_{i+1}$, and as above we conclude $x \notin \mathrm{Var}(\eta)$.

□

Finally, we can show that the set of definitions generated by Defs behave similarly to circuits, as long as the input formula only contains variables occurring in $\mathrm{Var}(\varphi)$.

Proposition 10. *Let η be an OPL formula with $\mathrm{Var}(\eta) \subseteq \mathrm{Var}(\varphi)$. Then, the set of definitions $\mathrm{Defs}(\eta)$ is a definition circuit over $\mathrm{Var}(\varphi)$.*

Proof. Straightforward from Propositions 6, 7, 8, and 9, once $\mathrm{Var}(\varphi) \subseteq V_0$ is taken into consideration. □

C.4 Circuits allow topological orderings

We now show that, in a circuit specified through a definition circuit over $\mathrm{Var}(\varphi)$, one can enumerate the internal nodes (i.e. the definitions) in such a way that nodes with an incoming edge (i.e. variables in the right-hand side of the definition) are either external nodes (i.e. variables in the original formula) or internal nodes occurring earlier in the enumeration. This will help us by using this same order to introduce overwrite rules, in such a way that after every

substring of overwrite rules the truth value for one of the extra variables is fixed depending on previous variables in the ordering.

Lemma 7. *Consider a strict subset of definitions $D \subset \mathit{Defs}(\varphi)$. Then, there exists a definition $x \leftrightarrow \delta$ in $\mathit{Defs}(\varphi) \setminus D$ such that, for all variables $y \in \mathrm{Var}(\delta) \setminus \mathrm{Var}(\varphi)$ some definition $y \leftrightarrow \delta'$ exists in D.*

Proof. We reason by contradiction. Assume such a definition $x \leftrightarrow \delta$ does not exist. We show by induction that we can construct a sequence of arbitrary length n Then, we construct an finite sequence $(x_i \leftrightarrow \delta_i)_{0 \leq i \leq n}$ where all definitions occur in $\mathrm{Defs}(\varphi) \setminus D$, and additionally $x_i \in \mathrm{Var}(\delta_{i-1})$ for all $1 \leq i \leq n$. In the base case $n = 0$, we know that $\mathrm{Defs}(\varphi) \setminus D$ is nonempty, so any $x_0 \leftrightarrow \delta_0$ in this set satisfies the conditions.

Now let us assume we have constructed a sequence $(x_i \leftrightarrow \delta_i)_{0 \leq i \leq n}$ in the conditions of the claim. Since the definition $x_n \leftrightarrow \delta_n$ is in $\mathrm{Defs}(\varphi) \setminus D$ and no definition satisfying the claim of the lemma exists, then there must be some variable $x_{n+1} \in \mathrm{Var}(\delta) \setminus \mathrm{Var}(\varphi)$ such that no definition of the form $x_{n+1} \leftrightarrow \delta$ occurs in D. Now, $\mathrm{Defs}(\varphi)$ is a definition circuit over $\mathrm{Var}(\varphi)$, so it is determined by $\mathrm{Var}(\varphi)$. Hence we conclude that there is a definition of the form $x_{n+1} \leftrightarrow \delta_{n+1}$ in $\mathrm{Defs}(\varphi) \setminus D$. Then, the sequence $(x_i \leftrightarrow \delta_i)_{0 \leq i \leq n+1}$ satisfies the induction claim.

Hence, by induction, we conclude that finite sequences as in the induction claim exists for any arbitrary length. Consider constructed the finite sequence $(x_i \leftrightarrow \delta_i)_{0 \leq i \leq n}$ for $n = \#\mathrm{Defs}(\varphi)+1$. By the pigeonhole principle, there exist indices $0 \leq i < j \leq n+1$ such that the definitions $x_i \leftrightarrow \delta_i$ and $x_j \leftrightarrow \delta_j$ coincide. Now, consider $y_k = x_{j-k}$ and $\gamma_k = \delta_{j-k}$ for $0 \leq k \leq j-i-1$. The resulting sequence of definitions $(x_k \leftrightarrow \delta_k)_{0 \leq k \leq j-i-1}$ satisfies $x_{k-1} \in \mathrm{Var}(\delta_k)$ for all $1 \leq k \leq j-i-1$, and furthermore $x_{j-i-1} \in \delta_0$. But this situation is precluded by $\mathrm{Defs}(\varphi)$ being acyclic, which follows from Proposition 10. □

Proposition 11. *There exists an ordering $(x_i \leftrightarrow \delta_i)_{1 \leq i \leq n}$ of $\mathit{Defs}(\varphi)$ such that, for every $i, j \in \{1, \ldots, n\}$ such that $x_i \in \mathrm{Var}(\delta_j) \setminus \mathrm{Var}(\varphi)$, then $i < j$.*

Proof. We show that, for all $0 \leq i \leq n = \#\mathrm{Defs}(\varphi)$ there is a finite sequence of distinct definitions $(x_j \leftrightarrow \delta_j)_{1 \leq j \leq i}$ satisfying the condition in the claim. We proceed by induction. The base case $i = 0$ is trivial, since the sequence needs to be empty. Now assume that we have constructed such a (possibly empty) finite sequence $(x_j \leftrightarrow \delta_j)_{i \leq j \leq i}$ of some length $i < n$. Then, by Lemma 7, we can find a definition $x_{i+1} \leftrightarrow \delta_{i+1}$ in $\mathrm{Defs}(\varphi)$ such that this definition does not occur in the constructed sequence, and $\mathrm{Var}(\delta_{i+1}) \setminus \mathrm{Var}(\varphi) \subseteq \{x_1, \ldots, x_i\}$. It is straightforward to check that the claim holds for $(x_j \leftrightarrow \delta_j)_{i \leq j \leq i+1}$ as well. □

C.5 Rule strings from primitive definitions

We now move from definition circuits to overwrite connectives. Observe that all the definitions introduced by Defs have very specific forms. We will refer to these as *primitive definitions*; a list of the definitions we consider primitive is provided in Table 1.

Lemma 8. *$\mathit{Defs}(\varphi)$ is a finite set of primitive definitions.*

Proof. Checking that the forms of the definitions match can be done by inspection. The fact that the variable in the left-hand side does not occur on the right-hand side follows from Proposition 7. □

Each definition in Table 1 is accompanied by a CNF translation and a string of overwrite rules. The following result can be easily, and boringly, checked.

definition	CNF translation	rule string
$x \leftrightarrow \top$	$\{[+x]\}$	$(+x :- \langle\rangle)$
$x \leftrightarrow \bot$	$\{[-x]\}$	$(-x :- \langle\rangle)$
$x \leftrightarrow p$	$\{[-x, +p], [+x, -p]\}$	$(+x :- +p)(-x :- -p)$
$x \leftrightarrow \neg p$	$\{[-x, -p], [+x, +p]\}$	$(+x :- -p)(-x :- +p)$
$x \leftrightarrow p \vee q$	$\{[-x, +p, +q], [+x, -p], [+x, -q]\}$	$(+x :- +p)(+x :- +q)(-x :- -p, -q)$
$x \leftrightarrow p \wedge q$	$\{[-x, +p], [-x, +q], [+x, -p, -q]\}$	$(-x :- -p)(-x :- -q)(+x :- +p, +q)$
$x \leftrightarrow p \wedge \neg q$	$\{[-x, +p], [-x, -q], [+x, -p, +q]\}$	$(-x :- -p)(-x :- +q)(+x :- +p, -q)$

Table 1: Primitive translations with their corresponding equivalent CNF translations, where x, p and q are distinct propositional variables.

Proposition 12. *Let $x \leftrightarrow \delta$ be a primitive definition, and consider its CNF translation F and its associated rule string ε. Then, $(x \leftrightarrow \delta) \equiv F$ holds, and additionally for every interpretation I, we have that $I + \varepsilon \vDash x \leftrightarrow \delta$. Furthermore, both $|\varepsilon|$ and $|F|$ are bounded by a constant.*

C.6 Finally, the translation

Let us *finally* show the translation from OPL into UOCNF. Given the OPL formula φ, we can find an ordering $(x_i \leftrightarrow \delta_i)_{1 \leq i \leq n}$ of $\mathrm{Defs}(\varphi)$ such that, for every $i, j \in \{1, \ldots, n\}$ such that $x_i \in \mathrm{Var}(\delta_j) \setminus \mathrm{Var}(\varphi)$, then $i < j$. By Lemma 8, we know that all the definitions $x_i \leftrightarrow \delta_i$ are primitive, so we can obtain their associated rule strings $\vec{\varepsilon} = \varepsilon_1, \ldots, \varepsilon_n$. Before we continue, let us show one last lemma, which gives sense to the need for a topological ordering:

Lemma 9. *Let η be any OPL formula. Let l_i for $0 \leq i \leq n$ be literals whose underlying variables do not occur in $\mathrm{Var}(\eta)$, and consider a rule string $\varepsilon = (l_1 :- \psi_1) \ldots (l_n :- \psi_n)$. Then, for every interpretation I, we have $I \vDash \eta$ if and only if $I + \varepsilon \vDash \eta$.*

Proof. Straightforward from observing that I and $I + \vec{\varepsilon}$ coincide in all variables from $\mathrm{Var}(\eta)$. □

We now show the main result, from which the existence of a simulation will be extracted as a corollary.

Theorem 5. *Let φ be an OPL formula with $\mathrm{Var}(\varphi) \subseteq V_0$, and consider a rule string $\vec{\varepsilon}$ constructed from a topological ordering as above. Then, for every interpretation I, we have $I + \vec{\varepsilon} \vDash \mathsf{node}(\varphi)$ if and only if $I \vDash \varphi$.*

Proof. We first show that $I + \vec{\varepsilon}$ satisfies $\mathrm{Defs}(\varphi)$ for every interpretation I. To do so, we show by induction on i that

$$I + \varepsilon_1, \ldots, \varepsilon_i \vDash \bigwedge_{j=1}^{i} (x_j \leftrightarrow \delta_j)$$

The case $i = 0$ is straightforward. Now assume that the induction claim holds for i, and let us show it for $i + 1$. Because of the conditions on the definition ordering, we know that $x_{i+1} \notin \mathrm{Var}(\delta_j)$ for $0 \leq j \leq i$. Furthermore, since $\mathrm{Defs}(\varphi)$ is univalent, we know that $x_{i+1} \notin \{x_1, \ldots, x_i\}$. Now, ε_{i+1} only contains overwrite rules that overwrite exclusively the variable x_{i+1}. Hence, by Lemma 9 we conclude that

$$I + \varepsilon_1, \ldots, \varepsilon_i \varepsilon_{i+1} \vDash \bigwedge_{j=1}^{i} (x_j \leftrightarrow \delta_j)$$

Furthermore, by Lemma 12 we know that $I + \varepsilon_1, \ldots, \varepsilon_i\varepsilon_{i+1}$ satisfies $x_{i+1} \leftrightarrow \delta_{i+1}$, so the induction claim holds true. By induction, we conclude that

$$I + \varepsilon_1, \ldots, \varepsilon_n \vDash \bigwedge_{j=1}^{n} (x_j \leftrightarrow \delta_j) \equiv \mathrm{Defs}(\varphi)$$

holds true. Hence, by Lemma 5, we conclude that, for every interpretation I, we have $I + \vec{\varepsilon} \vDash \varphi$ if and only if $I + \vec{\varepsilon} \vDash \mathsf{node}(\varphi)$. Finally, observe that $\mathrm{Defs}(\varphi)$ respects $\mathrm{Var}(\varphi)$, so in particular $I + \vec{\varepsilon} \vDash \varphi$ if and only if $I \vDash \varphi$, as we wanted. □

Corollary 5. *Let φ be an OPL formula. Then, there exists a linearly-sized UOCNF formula $\nabla\vec{\varepsilon}.\, F$ such that $\varphi \equiv \nabla\vec{\varepsilon}.\, F$.*

Proof. Consider the rule string $\vec{\varepsilon}$ constructed as in Theorem 5. Then, it is apparent that $\varphi \equiv \nabla\vec{\varepsilon}.\, \mathsf{node}(\varphi) \equiv \nabla\vec{\varepsilon}.\, \{[+\mathsf{node}(\varphi)]\}$. The linear size of the latter is a straightforward consequence of Propositions 4 and 12. □

D PR introduction as a UOCNF rule

Corollaries 2, 3, and 4 follow straightforward from Theorem 4, which we show here.

Theorem 4. *Let F be a CNF formula and C a PR clause in F upon some cube Q. Then, for every interpretation I satisfying F, the interpretation $I + \left(Q :- \overline{C}\right)$ satisfies $F \cup \{C\}$.*

Proof. Let I be an interpretation satisfying F. The interpretation $I + \left(Q :- \overline{C}\right)$ satisfies the clause C: if I does, then $I \nvDash C$, and so we have $I + \left(Q :- \overline{C}\right) = I \vDash C$; and if I does not satisfy C, then $I \vDash \overline{C}$, and then the clause $I + \left(Q :- \overline{C}\right)$ trivially satisfies the cube Q, which entails C because they have at least one literal in common.

Now assume that $I + \left(Q :- \overline{C}\right)$ does not satisfy F. This means that I and $I + \left(Q :- \overline{C}\right)$ must be different interpretations, which can only happen if $I \vDash \overline{C}$, so $I \nvDash C$ and $I + \left(Q :- \overline{C}\right) = I + Q$.

Let D be a clause in $F|_{\overline{C}}$. Then, there is another clause $D' \in F$ which shares no literal with $\overline{C}$, and $D' = D \setminus C$. Now, I satisfies F, which means that it satisfies D', so we can find a literal $l \in D'$ satisfied by I. Furthermore, I does not satisfy C, so the satisfied literal l is not in C, which means that $l \in D$. Hence, every clause in $F|_{\overline{C}}$ is satisfied by F. Now, every clause in $F|_Q$ is a RUP in $F|_{\overline{C}}$, so we conclude that I satisfies the CNF formula $F|_Q$.

We found before that $I + Q$ does not satisfy F. Hence, we can find a clause $D \in F$ which is not satisfied by $I + Q$. In particular, it does not satisfy any of its subclauses, for example $D \setminus \overline{Q}$. Now, $I + Q \vDash Q$ trivially, so we know that D shares no literals with Q. This means that the subclause $D \setminus \overline{Q}$ occurs in $F|_Q$, so it is satisfied by I. Let l be a satisfied literal by I in $D \setminus \overline{Q}$. Because D shares no literals with Q, and because $l \notin \overline{Q}$, we know that the underlying variable of l is not in $\mathrm{Var}(Q)$, but then it is mapped to the same truth value by both I and $I + Q$. Hence, $I + Q$ satisfies D, which is a contradiction. □

E Simulating PR with a proof system over overwrite clauses

Proposition 2. *Let C be a PR clause in a CNF formula F upon a cube Q, and $D \in F$. Then, either of the following hold: $Q \vDash D$, or $\overline{C} \vDash D|_Q$, or $C \cup D|_Q$ is a RUP in F.*

Proof. Let C be a PR clause in F upon Q, and D be a clause in F. Let us assume that the former two alternatives do not hold. Then, by the definition of PR clause, $D|_Q$ is a RUP in $F|_{\overline{C}}$. Then, there is a sequence of unit propagations which, using clauses $D_1, \ldots, D_n$ in $F|_{\overline{C}}$, and assumming the literals in $\overline{D} \setminus Q$, propagates literals $l_1, \ldots, l_n$ respectively, and $l_n = \overline{k}$ for some assummed or propagated literal k.

Then, by definition of reduct, each of the clauses D_i in $F|_{\overline{C}}$ corresponds to a clause $D'_i \in F$ such that $\overline{C} \not\models D'_i$, and additionally $D_i = D'_i \setminus C$. Now observe that the clauses D'_i propagate exactly the same literals $l_1, \ldots, l_n$ under the assumptions $\overline{C} \cup \overline{D} \setminus Q$. Hence, $C \cup D \setminus \overline{Q} = C \cup D|_Q$ is a RUP in F. □